GOODWILLIE CALCULUS

ORGANIZED BY
JUSTIN NOEL AND RUI REIS
SPRING 2012

Form. As usual, the seminar will be structured into six afternoon sessions. This term the talks will take place at the Mathematical institute at the University of Bonn in Lipschitz-Saal between 14:00 and 17:30 on the dates listed below. During each session there will be two talks with an intermission for coffee, tea, and socializing. There will be $75+\varepsilon$ minutes allocated to each talk.

Aaron Mazel-Gee has made his live-tex notes available online here:

- `http://math.berkeley.edu/~aaron/livetex/goodwillie.pdf`.

He has also requested that you email him (aaron@math.berkeley.edu) with any errors that you discover.

INTRODUCTION

Goodwillie's calculus of functors provides a sequence of conceptual and computational tools allowing one to use the language of calculus, normally used to analyze smooth functions, to instead analyze functors between topological categories.

Using this theory we can ask questions such as: What is the closest 'linear' approximation to the identity functor on pointed topological spaces? The answer turns out to be the functor

$$Q : X \mapsto \Omega^\infty \Sigma^\infty X.$$

The homotopy groups of QX are the stable homotopy groups of X. This functor is the closest approximation to the identity functor that is *excisive*, i.e., it takes homotopy pushouts to homotopy pullbacks, and serves as our definition of linear.

We then define degree n-polynomial functors using a generalization of this notion called *n-excisive*. The various polynomial approximations of a functor fit together into a tower, the *Taylor tower* of the functor. Just as in ordinary calculus, different functors have different radii of convergence, encoded in Goodwillie's definition of a *k-analytic functor*. For example the identity on pointed topological spaces is 1-analytic, i.e., the Taylor tower converges to the identity provided we restrict to connected spaces.

When the Taylor tower converges we obtain a spectral sequence associated to this tower. The input into this spectral sequences is the homotopy of the nth homogeneous approximation of the original functor. One of the most remarkable properties of Goodwillie calculus is that this approximation is infinitely deloopable[1] and that there is a *spectrum*, only dependent on the functor and n, called the *nth coefficient* of the Taylor series, which determines this infinite loop structure.

Returning to the example of the identity functor on pointed spaces, we obtain a spectral sequence converging to the unstable homotopy groups of connected space, whose input consists only of *stable* homotopy groups. One could interpret this as saying that unstable homotopy groups are naturally equipped with a filtration coming from stable phenomena. Arone-Mahowald showed that this filtration is closely

[1] Of course we must use the correct notion of 'spectrum' and 'infinite loop space' must be used when replace pointed topological spaces with a more general topological category

tied to the chromatic filtration [AM99]. This result can be used in combination with older computations to determine the v_1-periodic homotopy groups of spheres [Mah82, Tho90, MT92].

Computing the coefficients of the Taylor expansion of a functor is not easy, but thankfully not impossible either. The analysis involved is often deep and connected to other topics in mathematics. The potential payoffs to this work are enormous. Knowledge of the Taylor expansions of functors allows one to say things like: the closed linear approximation to the identity functor on augmented simplicial commutative rings is André-Quillen homology, or extremely deep statements like K-theory and topological cyclic homology differ by a locally constant functor [DGM10]. Which we use to reduce seemingly impossible tasks, i.e., the computation of $K_*(R)$ to a sequence of extraordinarily difficult tasks, i.e., the computation of $K_*(\mathbb{Z}), TC_*(\mathbb{Z}), TC_*(R)$ and the maps between them. Although the latter approach appears to be much more work, it has been fruitful and to the first authors limited knowledge, the primary method of computing higher algebraic K-theory.

PROGRAM

As general references we recommend [Kuh07a] (especially for newcomers), [Goo03, Goo10] and [Lur12, §7.1-7.2].

Remark 0.1. Each of the talks below will be marked according to difficulty:

- ♡ An elementary talk. The speaker will have well-written precise references to guide them through mostly standard material. The speaker will not need significant specialized knowledge to complete the talk, but examples may need to be worked out. A talk suitable for a beginning to intermediate graduate student.
- † An intermediate talk. The speaker will have well-written references to guide them through more specialized material. The speaker will not need significant specialized knowledge to complete the talk, but they might be required to judiciously summarize material. Such a talk is suitable for a more advanced graduate student or beyond.
- ‡ An advanced talk suitable for someone who has some familiarity with a related subject area or is willing to put in extra time into familiarizing themselves with the material.

Session 1: Overview and Applications
April 5th, 2012

Talk 1.1: Survey - Justin Noel

Discuss the original motivation for Goodwillie calculus with respect to the algebraic K-theory of spaces following [Goo91]. Describe the dictionary between ordinary calculus and Goodwillie calculus including:

- Functions $\leftrightarrow$ Functors
- Affine linear functions $\leftrightarrow$ Excisive functors
- Polynomials of degree $n \leftrightarrow n$-Excisive functors
- Maclaurin series expansion $\leftrightarrow$ Taylor tower at a point
- Taylor series expansion $\leftrightarrow$ Taylor tower at Y
- (For spaces) $c_n/n!x^n \leftrightarrow \Omega^\infty(C_n \wedge X^n)_{h\Sigma_n}$
- Radius of convergence of analytic function $\leftrightarrow$ Convergence of Taylor tower under connectivity hypothesis
- Faa di Bruno formula for $\partial_*(f\circ g)(0) \leftrightarrow \partial_*(F\circ G) = \partial_* F \circ \partial_* G$ (this is correct for endofunctors of spectra, general case is more complicated)

Given a Taylor tower for a q-analytic functor F, say landing in spaces or spectra, and a suitably q-connected X there is a spectral sequence converging to $\pi_* FX$ whose input is a bunch of *stable* homotopy groups. Some possible examples include the identity functor on (augmented) simplicial commutative rings, or commutative S-algebras. Or possibly describe the linear approximations to some of your favorite functors. For example, the difference between K-theory and TC is locally constant.

References: [Goo03, Goo10, Kuh07a, Lur12].

Talk 1.2: First derivatives and basic examples - Tibor Macko

The goal of this talk is to give an overview of [Goo90]. In particular the definition of the first derivative and the differential, i.e., the 1-jet, should be given. The remainder of the talk should be spent on examples of derivatives and differentials, namely the identity functor (and the other small examples discussed by Goodwillie), the stable function space functor, and the pseudoisotopy functor (without going into too much detail, since the details will appear in another talk). It would be nice to hear about the geometric motivation for studying A-theory as well as its relationship to pseudoisotopy theory, this is covered, in part, in the introduction to [DGM10].

Discuss the first derivatives of a multivariable functor [Goo03, 1.22],[Lur12, §7.1.2-7.1.1]. This is only a slight generalization of the one variable case, yet such derivatives will play a role in the construction of higher derivatives.

Session 2: Foundations I
April 19th, 2012

Talk 2.1: Homotopy (co)limits and n-excisive functors - Aaron Mazel-Gee

This talk should contain remind the audience about the homotopy (co)limits of cubical diagrams necessary for defining and proving the basic properties of the calculus derivatives and of the Taylor tower. The speaker may choose whether to work in the framework of simplicial model categories or in the framework of ∞ categories, but should mention that one can work in either framework. For the latter approach consult the relevant lecture from the previous AG on higher algebra. In either setting we need the notions of (strongly) cartesian and cocartesian diagrams and directed homotopy colimits and inverse limits.

Define and construct homotopy colimits and limits for cubical diagrams and directed diagrams. We are interested in those cubical diagrams which arise from the join construction in the sense of the beginning of [Goo03, §1]. Use the Bousfield-Kan formulas or the two sided (co)bar construction to construct these in any bicomplete simplicial model category. It is helpful to work through the case of homotopy cartesian and cocartesian diagrams in spaces explicitly.

Define n-excisive functors. Show Σ^∞ and $\Omega^\infty\Sigma^\infty$ are excisive. Some of the types of results one would like to know for studying convergence properties in Goodwillie calculus can be found in [Goo92, §1-2].

Extra Credit.

- Show $\Sigma^\infty X^m$ is m-excisive.
- Recall that every combinatorial model category is equivalent to an accessible localization of a category of simplicial presheaves [Dug01], [Lur09, 5.5.1.1]. Show, by reducing to the case of simplicial sets, that if this localization is accessible and left exact, then in such a category finite homotopy limits commute with directed homotopy colimits [Lur09, §5.3,Examples 7.3.4.4-7.3.4.7]. Note that such categories correspond precisely to ∞-topoi [Lur09, 6.1.0.4].

The existence of such limits and colimits, together with last property is essential for many arguments. For this reason Lurie calls such categories differentiable [Lur12, Definition 7.1.1.6].

- Prove, or at least discuss, the generalized Blakers-Massey theorem for topological spaces [Goo92, Props. 2.3-2.4]. It is advisable to stick to the case of square and (actual) cubical diagrams. This is the ur-example of results of this type, where we replace Id_{Top_*} with some other functor and category, typically depend on the original result. The Blakers-Massey theorem immediately implies that the identity functor is 1-analytic [Goo92, 4.2-4.3]. It also implies the Freudenthal suspension theorem, which in turn implies that Σ^∞ is 0-analytic (and linear) [Goo92, 4.4]. We can apply BM again to show $\Sigma^\infty_+ Top(K,-)$ is $\dim K$-analytic [Goo92, 4.5], which implies $(\Sigma^\infty_+ X)^{\wedge n}$ is 0-analytic [Goo03, 4.4]. BM also appears several times (amongst many other things) in the proof that $A(-)$ is 1-analytic. Finally, the proof of the generalized Blakers-Massey theorem for simplicial algebraic theories reduces to the result for simplicial sets, i.e., 'spaces' (see [Sch01, 3.6] for the ungeneralized form of BM).

References: A nice place to start is [Sad09]. A wonderful resource for learning about homotopy colimits is [Dug09]. I would try to understand the beginning of [Goo03] before turning to [Goo92].

Talk 2.2: The Taylor tower - Eric Peterson

Define the functors T_n and P_n and list their basic properties [Goo03, §1] or [Lur12, §7.1.1]. Namely these functors commute with filtered homotopy colimits and finite homotopy limits. In particular they commute with each other, they preserve fiber sequences of functors [Goo03, Props. 1.7 1.18], they commute with suspension, and [Goo03, 1.1+1.14] $P_n P_{n+k} \simeq P_n$. For the last result you will need to show that $P_n F$ is n-excisive [Goo03, Thm. 1.8] or [Lur12, 7.1.1.33]. Note that [Rez08] provides a streamlined proof of [Goo03, Lemma 1.9]. Rezk's argument is presented in the ∞-category context as [Lur12, Lemma 7.1.1.26]. To make sure that you have time, you might want to state this result and return to the proof at the end.

It would be good to mention that by the connectivity properties of T_n, the construction of $P_n F$ depends only on F restricted to very-connected spaces [Goo03, 1.1], so the Taylor tower only depends on the 'local' behaviour of F.

Define k-analyticity [Goo92, §4] if it has not already been presented. Use these results to construct the Taylor tower [Kuh07a, §5], [Goo03, Thm. 1.13]. Compute the first derivative of the identity functor on pointed spaces and all of the derivatives of the identity functor on spectra.

Extra Credit.

- Note that the rapidly increasing connectivity of the Taylor for an analytic functor F to spaces, spectra, or some category with a notion of homotopy groups (used to define connectivity), guarantees a strongly convergent spectral sequence for computing $\pi_* F(Y)$ naturally in suitably connected Y. More surprisingly if the functor lands in spectra we also obtain a strongly convergent spectral sequence computing the homology of $F(Y)$ from the homology of the ith derivatives with respect to a connective spectrum. We also obtain a strongly convergent spectral sequence for computing the ordinary cohomology of $F(Y)$ from the cohomology of the derivatives.
- Compute the 1st derivative of the identity functor on augmented commutative S-algebras [Kuh07a, 6.3] or $\Omega^\infty\Sigma^\infty$ on spectra.
- Use [Kuh07b, 6.3] to show that the first derivative $\partial Id(*)$ of the identity functor on commutative S-algebras (no augmentation) is trivial. In fact, all

of the derivatives at the terminal object are trivial, although the functor has non-trivial derivatives elsewhere.

- Reinterpret the main results of [Sch01] to characterize the 1st derivative of the identity functor on the category of pointed simplicial T-algebras for a simplicial algebraic theory T.
- Mention the role of the generalized Blakers-Massey theorem in showing that a functor, such as the identity functor on Top_*, is k-analytic.

References: [Kuh07a], [Goo03, §1], [Lur12, §7.1.1-7.1.2], and [Goo92].

Session 3: Foundations II
May 3rd, 2012

Talk 3.1: Derivatives are infinite loop spaces (The norm map) - Lennart Meier

Sketch the proof that the derivatives of reduced endofunctors of Top_* are infinite loop spaces [Goo03, Thm. 2.1]. A generalization of this result [Lur12, 7.1.2.9] appears and is proven in [Lur12, §7.1.2], which the speaker can follow if they are so inclined. It follows that for functors preserving homotopy colimits the nth differential $D^{(n)}F$, is determined by the nth coefficient $\partial_n F$ [Goo03, §5]. Since the proof of the above result, specifically the part about showing RF is $n-1$-excisive, is a bit technical, the speaker may want to instead move onto a somewhat unrelated topic, the norm map.

Optional: The norm map [Lur12, §7.1.6] is used to construct Tate cohomology which measures the failure of a G-spectrum to be free. For our purposes Tate spectra are important because, if F is a functor between two stable categories then P_nF can be constructed as a pullback of a map from $P_{n-1}F$ to a certain Tate spectrum [Kuh07a, p.15],[McC01], [Lur12, 7.1.6.29]. Since Kuhn showed that Tate cohomology lowers chromatic filtration [Kuh04], one can see that the Tate tower of an endofunctor of spectra splits $K(n)$-locally, leading to a number of applications [Kuh07a, §7]. These applications will partially be covered in a later talk.

Extra Credit. Discuss the derivatives in the relative case. It seems likely that a more detailed account of the theory presented in [Goo03] would use parametrized spectra.

Notice that reduced functors from Top_* naturally define prespectra by evaluation on spheres. There is an obvious extension of this notion to multivariable functors that are reduced in each variable. Restricting such a functor to the diagonal naturally gives a Σ_n prespectrum. Note that we can deloop the 0th space of the corresponding spectrum with respect to the standard (permutation) representation of Σ_n. In Schwede's terminology this means the associated spectrum is actually a fibrant Σ_n spectrum indexed over the 'natural' universe.

References: [Goo03]. For the norm map one could consult [Lur12, §7.1.6], which provides a lot of motivation but is consequently a little long.

Talk 3.2: Homogeneous functors and cross-effects - David Carchedi

The nth homogeneous approximation of F D_nF, as defined in the previous session, is the fiber of the canonical map $P_nF \to P_{n-1}F$. If we view the Taylor tower as a type of Postnikov tower for functors, then the homogeneous approximations correspond to the Eilenberg-MacLane spaces. Correspondingly understanding these functors is the first step in trying to understand the Taylor tower as a whole and their homotopical invariants serve as the input into spectral sequences coming from the Taylor tower.

There are several different ways to understand D_nF, each of which has an analogue in the world of functions. Firstly, in analogy with our definition above, we could think of it as a degree n homogeneous polynomial, i.e., a function of the form $f(x) = ax^n$, which is obtained as the difference of the degree n polynomial approximation from the degree $n-1$ polynomial approximation. Secondly, we could use the correspondence between polynomials of the form ax^n with symmetric multilinear (rational) functions via the cross-effect. This is a higher degree generalization of the correspondence between quadratic forms and symmetric bilinear forms

$$q(x) = ax^2 \mapsto s(x,y) = a(x+y)^2 - ax^2 - ay^2$$
$$s(x,y) \mapsto q(x) = s(x,x)/2!$$

In functor calculus we have a corresponding notion of a symmetric multilinear functor and we have a similar bijection which uses the cross-effect for functors. The role of dividing by $n!$ is now played by $(-)_{h\Sigma_n}$ the homotopy orbits functor. Finally, we can also understand such a function in terms of the coefficients appearing in the Taylor tower for f:

$$ax^n = (\partial^{(n)} f/\partial x^n)(0) \cdot x^n/n!.$$

In functor calculus, as will be explained in the following talk, the coefficients of the Taylor tower are come from stable objects, e.g., spectra.

The goal of this talk is to precisely explain these correspondences. In particular define (symmetric) multilinear functors and cross-effects [Lur12, §7.1.3-7.1.4]. It is useful to think of the cross-effect as simply the reduction of a multivariable functor. Give basic properties and examples (such as those coming from the previous talk). Especially important is the correspondence between n-homogeneous functors and symmetric n-multilinear functors [Lur12, 7.1.4.7] [Goo03, 3.5-3.6] (3.6 uses the correspondence from the previous talk). Show how one computes a higher derivative as iterated first derivatives of the cross-effect [Lur12, Prop. 7.1.3.23].

For simplicity the speaker can restrict to the case of reduced functors between pointed topological spaces/spectra and pointed topological spaces/spectra.

Extra Credit. Discuss what happens when one considers non-reduced functors. Although this introduces new technical difficulties, such as working with parametrized spectra, it is important for applications such as K-theory and A theory. The added generality is required for analytic continuation which tells us that if two analytic functors have the same first derivatives 'everywhere' then they differ by a constant.

References: [Kuh07a, §5], [Goo03, 2-6], and [Lur12, §7.1.2-7.1.4].

Session 4: Topological examples
May 24th, 2012

Talk 4.1: $\partial_*(\Sigma^\infty Top_*(K,-))(*)$ - Karol Szumilo

The goal of this talk is to cover the material in [Kuh07a, §6]. Arone [Aro99] gave a model for the entire Taylor tower of the first functor, showing

$$P_n \Sigma^\infty Top_*(K,X) = Spectra^{\mathscr{E}_{\leq n}}(K^{\wedge \cdot}, X^{\wedge \cdot})$$
$$\partial_d(*) = D(K^{(d)}).$$

Here $\mathscr{E}_{\leq n}$ is the subcategory of finite sets of cardinality at most n and surjections and $K^{(d)}$ is $K^{\wedge d}$ modulo the fat diagonal. A detailed account of this is provided in [AK02]. Special cases of interest include when K is an n-sphere and/or when X is an n-fold suspension. In the former cases the dth partial derivative can be realized as the stabilization of the configuration space of d-little n-discs (up to a shift). In the latter case we can see that the Taylor tower splits which is essentially equivalent to Snaith splitting. These results are sketched out in [Kuh07a, §6.1].

Show that all of the derivatives of $\Sigma^\infty\Omega^\infty$ are spheres following [Kuh07a, §6.2].

Talk 4.2: $\partial_*(Id_{Top_*})(*)$ - Irakli Patchkoria

Summarize and state the many results of Johnson [Joh95], Arone-Kankaanrinta [AK98], and Arone-Mahowald [AM99, 2.1] following [Kuh07a] on the derivatives of the identity functor on pointed spaces. Perhaps the best presentation of the computation of these derivatives is [AK98]. Give their computation of the derivatives modulo the fact about rigidifying their cosimplicial diagram. Alternative presentations can be found in [Goo03, §7] and [Lur12, §7.3.3].

For statements about the cohomology of the derivatives, consult [AD01] and [AM99].

Session 5: Dundas-Goodwillie-McCarthy
June 21st, 2012

Talk 5.1: TC and the cyclotomic trace - Steffen Sagave

The goal of this talk is to define the cyclotomic trace map. First define THH, topological Hochschild homology. Then show that THH receives a canonical map from K-theory, the Dennis trace map, first constructed [Bok90] in the context of functors with smash product. Then using *edgewise subdivision*, to construct the cyclotomic structure on THH, construct TC, and then lift the Dennis trace map to the *cyclotomic* trace map.

To move through the ambitious list of tasks above, the speaker should feel free to use whatever foundations are convenient and to skip all technical arguments. Blumberg and Mandell [BM10] is probably the most modern treatment of this material and [DGM10] the most expansive. Although both of these make use of the generality of generalized rings (small spectral or additive categories). There is a nice presentation in the context of symmetric ring spectra in [Sch09]. Following Bokstedt's construction of the Dennis trace [Bok90], was the generalization of this construction to ring functors and to include TC in [DM96].

Since there is a lot to go through in this talk, the speaker should work with the following speaker in order to cover this material.

References: [DGM10], [Mad94], [Goo91], [DM94], [Dun97], [DM96], [McC97].

Talk 5.2: The comparison between K and TC - Jeremiah Heller

The speaker should coordinate with the previous speaker to see if they will need extra time or to possibly take over some of the responsibilities of that talk.

The goal of this talk is to sketch the proof that the cyclotomic trace induces an equivalence of first derivatives $\partial_x K \to \partial_x TC$ or, equivalently, that the difference between algebraic K-theory and topological cyclic homology is locally constant. The motivation behind this talk is Goodwillie's conjecture [Goo91] that for any 1-connected map of S-algebras $B \to A$ the square

$$\begin{array}{ccc} K(B) & \xrightarrow{Trc} & TC(B) \\ \downarrow & & \downarrow \\ K(A) & \xrightarrow{Trc} & TC(A) \end{array}$$

is homotopy cartesian. The goal of this talk is to give an overview of the proof of this conjecture following [DGM10, VII.1] in the more general case of a map of S-algebras for which $\pi_0(B) \to \pi_0(A)$ is a surjection with nilpotent kernel.

Extra Credit. Mention the results of [BCC$^+$96] where they use trace methods to prove the algebraic K-theory analog of the Novikov conjecture.

References: [DGM10], [Goo91], [Mad94], [BCC$^+$96].

Session 6: Derivatives of Waldhausen's $A(X)$ and chromatic homotopy theory

July 5th, 2012

Talk 6.1: $\partial_* A(-)$ - Wolfgang Steimle

The goal of this talk is to compute the derivatives of Waldhausen's algebraic K-theory of spaces functor. The talk should describe, in as much detail as time allows (if it has not already been covered in the first session), the first derivative of the pseuso-isotopy functor of [Goo90], and as a corollary obtain the first derivative of $A(X)$.

The description of the higher derivatives is made using a *trace* map $\tau : A(X) \to L(X)$ whose target is the functor $L(X) = \Sigma^\infty_+ Top(S^1, X)$ [Goo91], the unreduced suspension spectrum of the free loop space. Time permitting this map should be defined and perhaps some of its basic properties discussed. The derivatives of $L(X)$ recalled [Goo03, 7.1]. One should describe the first derivative of $A(X)$ using the trace map and then finish by describing the higher derivatives of this functor [Goo03, 9.7].

Talk 6.2: Interactions with chromatic homotopy theory - Markus Szymik

Present selected results from [Kuh07a, §7]. One should spend the first 20-30 minutes of the talk reminding the audience about the fundamental results about telescopic/$K(n)$-localization and the chromatic filtration [Rav84, Rav90].

Expand on the discussion of the derivatives of Id_{Top_*} and why the Taylor tower splits after telescopic localization [Kuh04]. Time permitting it would be nice to hear about the computations of v_1-periodic homotopy groups of odd spheres [MT92, Mah82]. Alternatively one could mention Kuhn's computations of the Morava K-theory of infinite loop spaces [Kuh06].

References

[AD01] G. Z. Arone and W. G. Dwyer, *Partition complexes, Tits buildings and symmetric products*, Proc. London Math. Soc. (3) **82** (2001), no. 1, 229–256. MR MR1794263 (2002d:55003)

[AK98] Greg Arone and Marja Kankaanrinta, *A functorial model for iterated Snaith splitting with applications to calculus of functors*, Stable and unstable homotopy (Toronto, ON, 1996), Fields Inst. Commun., vol. 19, Amer. Math. Soc., Providence, RI, 1998, pp. 1–30. MR MR1622334 (99b:55009)

[AK02] Stephen T. Ahearn and Nicholas J. Kuhn, *Product and other fine structure in polynomial resolutions of mapping spaces*, Algebr. Geom. Topol. **2** (2002), 591–647 (electronic). MR MR1917068 (2003j:55009)

[AM99] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788. MR MR1669268 (2000e:55012)

[Aro99] Greg Arone, *A generalization of Snaith-type filtration*, Trans. Amer. Math. Soc. **351** (1999), no. 3, 1123–1150. MR MR1638238 (99i:55011)

[BCC+96] M. Bökstedt, G. Carlsson, R. Cohen, T. Goodwillie, W. C. Hsiang, and I. Madsen, *On the algebraic K-theory of simply connected spaces*, Duke Math. J. **84** (1996), no. 3, 541–563. MR MR1408537 (97h:19002)

[BM10] A. J. Blumberg and Mandell M., *Localization theorems in topological Hochschild homology and topological cyclic homologyy*, To appear (2010).

[Bok90] M. Bokstedt, *Topological Hochschild homology*, Preprint (1990).

[DGM10] B. Dundas, T. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*, `http://www.uib.no/People/nmabd/b/b.pdf`, 2010.

[DM94] Bjørn Ian Dundas and Randy McCarthy, *Stable K-theory and topological Hochschild homology*, Ann. of Math. (2) **140** (1994), no. 3, 685–701. MR 1307900 (96e:19005a)

[DM96] ———, *Topological Hochschild homology of ring functors and exact categories*, J. Pure Appl. Algebra **109** (1996), no. 3, 231–294. MR 1388700 (97i:19001)

[Dug01] Daniel Dugger, *Combinatorial model categories have presentations*, Adv. Math. **164** (2001), no. 1, 177–201. MR 1870516 (2002k:18022)

[Dug09] D. Dugger, *A primer on homotopy colimits.*, `http://pages.uoregon.edu/ddugger/hocolim.pdf`, 2009.

[Dun97] Bjørn Ian Dundas, *Relative K-theory and topological cyclic homology*, Acta Math. **179** (1997), no. 2, 223–242. MR 1607556 (99e:19007)

[Goo90] Thomas G. Goodwillie, *Calculus. I. The first derivative of pseudoisotopy theory*, K-Theory **4** (1990), no. 1, 1–27. MR 1076523 (92m:57027)

[Goo91] ———, *The differential calculus of homotopy functors*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 621–630. MR 1159249 (93g:55015)

[Goo03] ———, *Calculus. III. Taylor series*, Geom. Topol. **7** (2003), 645–711 (electronic). MR 2026544 (2005e:55015)

[Goo10] T. Goodwillie, *Notes from london mathematical society invited lecture.*, 2010.

[Goo92] Thomas G. Goodwillie, *Calculus. II. Analytic functors*, K-Theory **5** (1991/92), no. 4, 295–332. MR 1162445 (93i:55015)

[Joh95] Brenda Johnson, *The derivatives of homotopy theory*, Trans. Amer. Math. Soc. **347** (1995), no. 4, 1295–1321. MR 1297532 (96b:55012)

[Kuh04] Nicholas J. Kuhn, *Tate cohomology and periodic localization of polynomial functors*, Invent. Math. **157** (2004), no. 2, 345–370. MR MR2076926 (2005f:55008)

[Kuh06] ———, *Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces*, Adv. Math. **201** (2006), no. 2, 318–378. MR MR2211532 (2007d:55006)

[Kuh07a] ———, *Goodwillie towers and chromatic homotopy: an overview*, Proceedings of the Nishida Fest (Kinosaki 2003), Geom. Topol. Monogr., vol. 10, Geom. Topol. Publ., Coventry, 2007, pp. 245–279. MR MR2402789

[Kuh07b] ———, *Primitives and central detection numbers in group cohomology*, Adv. Math. **216** (2007), no. 1, 387–442. MR MR2353262 (2008i:20061)

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR MR2522659

[Lur12] J. Lurie, *Higher algebra*, Available at `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`, 2012.

[Mad94] Ib Madsen, *The cyclotomic trace in algebraic K-theory*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel, 1994, pp. 213–241. MR MR1341845 (96f:55005)

[Mah82] Mark Mahowald, *The image of J in the EHP sequence*, Ann. of Math. (2) **116** (1982), no. 1, 65–112. MR MR662118 (83i:55019)

[McC97] Randy McCarthy, *Relative algebraic K-theory and topological cyclic homology*, Acta Math. **179** (1997), no. 2, 197–222. MR 1607555 (99e:19006)

[McC01] ———, *Dual calculus for functors to spectra*, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 183–215. MR 1831354 (2002c:18009)

[MT92] Mark Mahowald and Robert D. Thompson, *The K-theory localization of an unstable sphere*, Topology **31** (1992), no. 1, 133–141. MR MR1153241 (92k:55017)

[Rav84] Douglas C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), no. 2, 351–414. MR MR737778 (85k:55009)

[Rav90] ———, *The nilpotence and periodicity theorems in stable homotopy theory*, Astérisque (1990), no. 189-190, Exp. No. 728, 399–428, Séminaire Bourbaki, Vol. 1989/90. MR MR1099883 (92g:55013)

[Rez08] C. Rezk, *A streamlined proof of Goodwillie's n-excisive approximation*, December 2008.

[Sad09] H. Sadofsky, *Goodwillie calculus notes*, `http://noether.uoregon.edu/~sadofsky/gctt/goodwillie.pdf`, April 2009.

[Sch01] S. Schwede, *Stable homotopy of algebraic theories*, Topology **40** (2001), no. 1, 1–41. MR MR1791267 (2002d:55026)

[Sch09] Christian Schlichtkrull, *The cyclotomic trace for symmetric ring spectra*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 545–592. MR 2544396 (2011b:19004)

[Tho90] Robert D. Thompson, *The v_1-periodic homotopy groups of an unstable sphere at odd primes*, Trans. Amer. Math. Soc. **319** (1990), no. 2, 535–559. MR 1010890 (90j:55021)