EQUIVARIANT HOMOTOPY THEORY: PROBLEM SET 1

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(1) Suggested reading [DS95] and [Hov99, Ch. 1].
(2) Show that if $\mathscr{C}$ is a model category then $\mathscr{C}^{\mathrm{op}}$ is a model category.
(3) Suppose f is a cofibration in a model category $\mathscr{C}$ and the following diagram is a pushout:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

Show that g is a cofibration. State and prove the dual form of this result using the previous exercise.
(4) Show that a Serre fibration (see [Hov99, §2.4]) $f\colon E \to B$ with fiber $i\colon F \to E$ and B gives rise to a long exact sequence

$$\cdots \to \pi_{n+1}(B, fi(x)) \xrightarrow{\partial} \pi_n(F, x) \xrightarrow{i} \pi_n(E, i(x)) \xrightarrow{f} \pi_n(B, fi(x)) \to \cdots \to \pi_0(E, i(x)) \to \pi_0(B, fi(x)).$$

Here $x \in F$ is an arbitrary basepoint. Hint: use the long exact sequence of a pair $F \to E$ of spaces, if you have not seen this long exact sequence before, you can find it in standard sources [Hat02, May99], although you should try to construct it yourself.
(5) Show that every set X is $|X|$-small. Hint: you must use the definition of $|X|$-filtered ordinal in an essential and obvious way.
(6) Find an example of a space X (necessarily not locally compact Hausdorff) such that the functor $X \times -$ does not preserve colimits.

References

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. MR MR1361887 (96h:55014)

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999. MR 1650134 (99h:55031)

[May99] J. P. May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999. MR MR1702278 (2000h:55002)