EQUIVARIANT HOMOTOPY THEORY: PROBLEM SET 3

JUSTIN NOEL

(1) Suggested reading [DS95], the beginning of [Qui67], and [Hov99, §2.1, §2.4]. Also [Str09] for CGWH spaces.

(2) Suppose that $U\colon \mathscr{D} \to \mathscr{C}$ admits a right adjoint between two bicomplete categories. Moreover, suppose that U creates filtered colimits and all limits and that $\mathscr{C}$ admits a cofibrantly generated model structure. Say a morphism f in $\mathscr{D}$ is a weak equivalence/fibration if Uf is a weak equivalence/fibration. Use [Hov99, Thm. 2.1.19] to give conditions that guarantee that U *right induces* a cofibrantly generated model structure on $\mathscr{D}$. That is it has a cofibrantly generated model structure such that the weak equivalences and fibrations are those above.

(3) Let $\mathscr{C}$ be a cofibrantly generated model category and X be an object of $\mathscr{C}$. If $\mathscr{C}_{/X}$ is the category of objects under X (and morphisms commuting triangles), show that the forgetful functor $U\colon \mathscr{C}_{/X} \to \mathscr{C}$ right induces a cofibrantly generated model structure on $\mathscr{C}_{/X}$.

(4) Let $s\mathcal{G}roup = \mathcal{G}roup^{\Delta^{\mathrm{op}}}$ be the category of simplicial groups and $U\colon s\mathcal{G}roup \to s\mathcal{S}et$ be the forgetful functor.
   - (i) Construct a left adjoint to U.
   - (ii) Show that U creates all limits and filtered colimits.
   - (iii) Use the above exercise, Quillen's model structure on simplicial sets, the fact that all simplicial sets are small, and all simplicial sets that are simplicial groups are fibrant to see that U right induces a model structure on $s\mathcal{G}roup$.
   - (iv) What are the generating cofibrations and acyclic cofibrations?

(5) (Postponed to problem set 3) ([May99, Ch. 5] Show that
   - (i) Any subspace of a weak Hausdorff space is weak Hausdorff.
   - (ii) Any closed subspace of a k-space is a k-space.
   - (iii) An open subset U of a CGWH X is CGWH if each point $x \in U$ has an open neighborhood in X with closure contained in U.
   - (iv) A space is Tychonoff (points are closed, and for each point $x \in X$ and closed subset A not containing x there is a continuous function $f\colon X \to I$ such that $f(x) = 0$ and $f(A) = 1$) if and only if it can be embedded in a cube.
   - (v) There are Tychonoff spaces that are not k-spaces, but every cube is a compact Hausdorff space.
   - (vi) In view of the above, what should a subspace of CGWH-space be?

(6) Suppose that X is a non-empty space equipped with the indiscrete topology, then the w-ification of X is the one point space.

(7) Let $\mathbb{Q} \subset \mathbb{R}$ be equipped with the subspace topology. Show that the k-ification of $\mathbb{Q}$ is $\mathbb{Q}$ equipped with the discrete topology.

(8) Show that $\mathbb{R}/\mathbb{Q}$ is a point in compactly generated weak Hausdorff spaces.

(9) Suppose H is a topological subgroup of a compactly generated weak Hausdorff topological group G. Show that $G/H = G/\overline{H}$ in compactly generated weak Hausdorff spaces.

References

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