EQUIVARIANT HOMOTOPY THEORY: PROBLEM SET 6

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(1) Suggested reading: [May96, Ch. 16] for simplicial objects and the classifying space construction, [McC01, Ch. 8bis] for group (co)homology with twisted and untwisted coefficients.

(2) [May96, Ch. 16.1] Let X be a connected CW-complex, with $\pi = \pi_1(X)$, and universal cover $\widetilde{X}$ regarded as a π-space. Let M be an abelian group with trivial π-action. Show that $\widetilde{X}$ inherits the structure of a free π-CW complex and that

$$C_*(X;M) \cong M \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}).$$

(3) Bonus: Prove the analogue of the previous result for M a local coefficient system as well.

(4) [May96, Ch. 16.2] Use the first exercise to show

$$H_*(B\pi;M) \cong \operatorname{Tor}_*^{\mathbb{Z}[\pi]}(M,\mathbb{Z}).$$

(5) Show that $H^*(B\pi;M) \cong \operatorname{Ext}^*_{\mathbb{Z}[\pi]}(\mathbb{Z},M)$.

(6) Bonus: Use the previous bonus problem to show the analogue of the previous statement for twisted coefficients.

(7) When $\pi = C_n = \langle g \rangle$, use the periodic free resolution:

$$\cdots \to \mathbb{Z}[\pi] \xrightarrow{1-g} \mathbb{Z}[\pi] \xrightarrow{\sum g^i} \mathbb{Z}[\pi] \xrightarrow{1-g} \mathbb{Z}[\pi] \xrightarrow{\epsilon} \mathbb{Z}$$

to calculate the (co)homology (as a graded abelian group) with coefficients in $\mathbb{Z}$ and $\mathbb{F}_p$ for each prime p. Here ϵ is the unique $\mathbb{Z}[\pi]$-module map satisfying $\epsilon(g) = 1$.

(8) Bonus: Suppose that $2|n$. Let $\mathbb{Z}^{\pm}$ be $\mathbb{Z}$ with C_n acting by $g \cdot 1 = -1$. Use the previous bonus problem and the above resolution to calculate $H_*(BC_n;\mathbb{Z}^{\pm})$ and $H^*(BC_n;\mathbb{Z}^{\pm})$.

(9) Bonus: Use the Yoneda product in Ext ([Eis95, Exercise A.3.26-27]) and the periodicity of the resolution to find an exact sequence of length 2 representing a generator in $\operatorname{Ext}^2_{\mathbb{Z}[C_n]}(\mathbb{Z},\mathbb{Z})$. Show that under iterated products (given by gluing sequences) this gives a polynomial generator and that $H^*(BC_n;\mathbb{Z}) \cong \mathbb{Z}[x]/(nx)$.

(10) Bonus: What happens when we do the previous exercise with $\mathbb{F}_p$ coefficients? (Hint: Use the structure theorem for finitely generated abelian groups and the homeomorphism $B(G \times H) \cong BG \times BH$ to reduce to the case $G = C_{q^i}$ for some prime q).

(11) Bonus: Use the Serre spectral sequence for the fibration

$$S^1 \xrightarrow{p^n} S^1 \to BC_{p^n}$$

to calculate the ring structure $H^*(BC_{p^n};\mathbb{F}_p)$ again.

(12) Bonus: For abelian groups G, BG can be given the structure of a topological abelian group. This makes $H_*(BG;\mathbb{F}_p)$ into a graded commutative ring (in fact, a connected bicommutative Hopf algebra). When $G = C_{p^n}$, what is this ring?

(13) Bonus: What happens in the previous exercise if we use $\mathbb{Z}$ coefficients?

References

[Eis95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995.

[May96] J. P. May, *Equivariant homotopy and cohomology theory*, CBMS Regional Conference Series in Mathematics, vol. 91, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996, With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner. MR MR1413302 (97k:55016)

[McC01] John McCleary, *A user's guide to spectral sequences*, second ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. MR 1793722 (2002c:55027)