Quasicategorical Adjunctions

Talk by Tom Fiore
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Gewünschtes Beispiel Let
$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$
be a simplicial Quillen adjunction between simplicial model categories.

$\mathcal{C}^{o}$:= simplicially full sub simplicial category of $\mathcal{C}$ on the cofibrant-fibrant objects

Then there should be an adjunction of quasicats
$$\begin{tikzcd} N^{simp}(\mathcal{C}^{o}) \arrow[r, bend left] \arrow[r, phantom, "\perp"] & N^{simp}(\mathcal{D}^{o}) \arrow[l, bend left] \end{tikzcd}.$$

Problem F, G do not preserve cofibrant-fibrant objects, so we do not have F^{o} and G^{o}.

Outline We present Joyal's definition of adjunction via the 2-category $QCat_2$ of quasicategories, and Lurie's definition of adjunction as a Cartesian, coCartesian fibration $\mathcal{M} \longrightarrow \Delta[1]$.
Then we prove that left adjoints preserve colimits.

Joyal's 2-Category of Quasicategories and Joyal's Definition of Adjunction

$\underline{QCat} :=$ simp cat. of quasicategories
with $\underline{QCat}(X,Y) := Y^X \in SSet$.

$QCat_2 :=$ 2-cat. w/ objects quasicategories
and $QCat_2(X,Y) := ho(Y^X)$.

ho preserves products, so comp in $QCat_2$ comes from $Z^Y \times Y^X \longrightarrow Z^X$.

$Mor(QCat_2) =$ functors of quasicategories
$2\text{-}Cell(QCat_2) =$ homotopy classes of natural transf's.

Def (Joyal) An adjunction between quasicategories is an adjunction in the 2-category $QCat_2$, i.e.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

and $[\eta] : Id_X \Rightarrow g \circ f$ and $[\varepsilon] : f \circ g \Rightarrow id_Y$
such that the two triangle identities hold:

$$f \xRightarrow{i_f * [\eta]} fgf \xRightarrow{[\varepsilon] * i_f} f \quad = \quad i_f$$

$$g \xRightarrow{[\eta] * i_g} gfg \xRightarrow{i_g * [\varepsilon]} g \quad = \quad i_g \ .$$

$*$ means the "horizontal composition" of 2-cells in the 2-category $QCat_2$, as in any 2-category, $i_f :=$ identity 2-cell on morphism f

The triangle identities can also be drawn as

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[r, bend right=60, "f"'] \arrow[r, phantom, bend right=30, "\Downarrow i_f"] \arrow[rr, bend left=60, "\mathrm{id}_X"] \arrow[rr, phantom, bend left=30, "\Downarrow {[\eta]}"] & Y \arrow[r, "g"] \arrow[rr, bend right=60, "\mathrm{id}_Y"'] \arrow[rr, phantom, bend right=30, "{[\varepsilon]}"] & X \arrow[r, "f"'] \arrow[r, bend left=60, "f"] \arrow[r, phantom, bend left=30, "\Downarrow i_f"] & Y & {=} & X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "f"'] \arrow[r, phantom, "\Downarrow i_f"] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y \arrow[r, "g"'] \arrow[r, bend left=60, "g"] \arrow[r, phantom, bend left=30, "\Downarrow i_g"] \arrow[rr, bend right=60, "\mathrm{id}_Y"'] \arrow[rr, phantom, bend right=30, "\Downarrow {[\varepsilon]}"] & X \arrow[r, "f"'] \arrow[rr, bend left=60, "\mathrm{id}_X"] \arrow[rr, phantom, bend left=30, "\Downarrow {[\eta]}"] & Y \arrow[r, "g"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, bend right=30, "\Downarrow i_g"] & X & {=} & Y \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow i_g"] & X
\end{tikzcd}$$

.

Ex If F, G, η, ε are an adjunction of ordinary categories, then $NF, NG, [N\eta], [N\varepsilon]$ are an adjunction of quasicategories.

(Recall $N(\mathcal{C} \times [1]) = N\mathcal{C} \times \Delta[1]$)

Prop $QCat_2 \xrightarrow{ho} Cat$ is a 2-functor

Pr ho is $QCat_2(\Delta[0], -) : QCat_2 \longrightarrow Cat$, (Thanks Georgios!) so a 2-functor.

$$h_0(X) \cong h_0(X^{\Delta[0]}) = QCat_2(\Delta[0], X)$$

$\square$

<u>Cor</u> ho maps an adjunction to an adjunction.

<u>Rem</u> Adjunctions of quasicategories extend to homotopically coherent adjunctions in the following sense, as done by Riehl-Verity (2016).

<u>Notation</u> Δ_∞ := subcategory of Δ that has morphisms those simplicial operators that preserve the <u>top</u> elements

$\Delta_{-\infty}$:= subcategory of Δ that has morphisms those simplicial operators that preserve the <u>bottom</u> elements

Δ_+ := Δ with also the empty set as an object
= "free strict monoidal category containing a monoid"

<u>Adj</u> := simplical category obtained as the hom-wise nerve of the Schanuel-Street 2-category Adj of the free living adjunction (in Cahiers in 1986)

$\mathrm{Obj}(\mathrm{Adj}) = \{+,-\}$

$$\begin{tikzcd}
\mathrm{Adj}: \Delta_+^{op} \arrow[loop left] & - \arrow[r, bend right, "\Delta_\infty"'] & + \arrow[l, bend right, "\Delta_{-\infty}"'] \arrow[loop right] & \Delta_+
\end{tikzcd}$$

<u>Thm</u> (Schanuel-Street 1986) An adjunction in a 2-category $\mathcal{C}$ is the same as a 2-functor $\mathrm{Adj} \longrightarrow \mathcal{C}$.

<u>Thm</u> (Riehl-Verity 2016) Let $A: \mathrm{Adj} \longrightarrow \mathrm{QCat}_2$ be an adjunction of quasicategories. Then A extends to a homotopy coherent adjunction in the sense that there exists a simplicial functor $\underline{A}: \underline{\mathrm{Adj}} \longrightarrow \underline{\mathrm{QCat}}$ such that

$$\mathrm{Adj} \cong \mathrm{ho}(\underline{\mathrm{Adj}}) \xrightarrow{\mathrm{ho}(\underline{A})} \mathrm{ho}(\underline{\mathrm{QCat}}) = \mathrm{QCat}_2$$

is the original adjunction A.

(Here $\mathrm{ho}(\ )$ means to apply $\mathrm{ho}(\ )$ hom-wise, and $\mathrm{Adj} \cong \mathrm{ho}(\underline{\mathrm{Adj}})$ via the counit isomorphism for nerve).

Marked Simplicial Sets and Marked Anodyne Maps

To move towards Lurie's notion of adjunction, we take a detour to marked simplicial sets. This knowledge will also be useful when we discuss the Cartesian model structure on

$$\begin{gathered}\text{Marked simplicial sets over } S^{\#} \\ \| \\ (\mathrm{Set}_{\Delta}^{+})/S^{\#}\end{gathered}$$

for any simplicial set S.

Def A <u>marked simplicial set</u> $(S, \mathcal{E})$ is a simplicial set S and a collection $\mathcal{E}$ of edges which contains every degenerate edge. The edges in $\mathcal{E}$ are called <u>marked</u>.

<u>Notation</u> (i) If S is a simplicial set then $S^{\#} := (S, S_1)$ has <u>every</u> edge marked.

(ii) If S is a simplicial set, then $S^{\flat} := (S, s_0(S_0))$ has only the <u>degenerate</u> edges marked.

(iii) If $p: X \to S$ is a Cartesian fibration, then $X^{\natural} = (X, \mathcal{E})$ has only p-Cartesian edges of X marked.

Notation $\mathrm{Set}_\Delta^+ :=$ category of marked simplicial sets

A morphism $f : (S, \mathcal{E}) \longrightarrow (S', \mathcal{E}')$ is a simplicial set map $f : S \longrightarrow S'$ such that $f_1(\mathcal{E}) \subseteq \mathcal{E}'$.

Def The class of marked anodyne morphisms in Set_Δ^+ is the weakly saturated class generated by the following morphisms.

(i) $\forall\, 0<k<n$, $(\Lambda^k[n])^\flat \hookrightarrow (\Delta[n])^\flat$

[or equivalently $A^\flat \hookrightarrow B^\flat$ for every inner anodyne map $A \hookrightarrow B$ of simpl. sets]

(ii) $\forall\, n>0$,

$$(\Lambda^n[n], \mathcal{E} \cap (\Lambda^n[n])_1) \hookrightarrow (\Delta[n], \mathcal{E})$$

where $\mathcal{E} = \{(n-1) \to n\} \cup \{\text{deg. edges of } \Delta[n]\}$

(iii)

$$(\Lambda^1[2])^\# \coprod_{(\Lambda^1[2])^\flat} (\Delta[2])^\flat \hookrightarrow (\Delta[2])^\#$$

and

(iv) $\forall$ Kan complex K, the map $K^\flat \longrightarrow K^\#$.

<u>Prop</u> A morphism $(Y, \mathcal{E}) \longrightarrow S^\#$ in Set_Δ^+ has the right lifting property w.r.t. all marked anodyne maps if and only if the underlying map $Y \longrightarrow S$ is a Cartesian fibration and $(Y, \mathcal{E}) = Y^\natural$ (i.e. the marked edges $\mathcal{E}$ are precisely the p-Cartesian morphisms of Y).

<u>Rem</u> In particular, if $p : X \longrightarrow S$ is a Cartesian fibration, then every commutative square of the form

$$\begin{tikzcd} {} \arrow[r] \arrow[d, "\text{marked anodyne}"'] & X^\natural \arrow[d, "p"] \\ {} \arrow[r] & S^\# \end{tikzcd}$$

in Set_Δ^+ admits a diagonal filler.

<u>Prop</u> 3.1.2.3 If $f : X \to X'$ and $g : Y \longrightarrow Y'$ are morphisms in Set_Δ^+, f is marked anodyne, and the underlying map of g is a monomorphism, then $f \times' g : (X \times Y') \coprod_{X \times Y} (X' \times Y) \longrightarrow X' \times Y'$ is marked anodyne.

Cor If $f: X \to X'$ is marked anodyne, and Y' is any marked simplicial set, then $f \times id_{Y'} : X \times Y' \longrightarrow X' \times Y'$ is marked anodyne.

Pr Take $Y = \emptyset$ in previous proposition.

□

Cartesian Fibrations $p: \mathcal{M} \longrightarrow \Delta[1]$ and Functors

$$\begin{tikzcd} g : \mathcal{D} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{C} \arrow[d, phantom, "\cong" rotate=90] \\ p^{-1}(1) & p^{-1}(0) \end{tikzcd}$$

Recall Classically, for ordinary categories B, there is a strict 2-equivalence of 2-categories

$$(\text{Grothendieck Fibs over } B) \rightleftarrows (\text{Pseudo Functors } B^{op} \longrightarrow Cat).$$

If $p: M \longrightarrow B$ is a Grothendieck fibration of ordinary categories and $f: b_0 \to b_1$ is a morphism in the base B, then the functor

$f^* : p^{-1}(b_1) \longrightarrow p^{-1}(b_0)$ is defined on objects by: for $y \in p^{-1}(b_1)$, choose a p-Cartesian lift $\tilde{f}: x \longrightarrow y$ of f and set $f^* y := x$.

On morphisms, $f^*: p^{-1}(b_1) \to p^{-1}(b_0)$ is defined via the universal property of p-Cartesian morphisms and the already selected objects.

Rem The analogous story for Cartesian fibrations of simplicial sets is more involved. We only show how to construct $f^* = g$ for $\mathcal{M} \to \Delta[1]$ according to [HTT, Section 5.2.1].

Def 5.2.1.1 Let $p: \mathcal{M} \to \Delta[1]$ be a Cartesian fibration and $h_0: \mathcal{C} \to p^{-1}(0)$, $h_1: \mathcal{D} \to p^{-1}(1)$ equivalences of quasicategories. We say that $g: \mathcal{D} \to \mathcal{C}$ is associated to p, h_0, h_1 if there exists $s: \mathcal{D} \times \Delta[1] \to \mathcal{M}$ such that

(i)
$$\begin{tikzcd}
\mathcal{D} \times \Delta[1] \arrow[rr, "s"] \arrow[dr, "pr_2"'] & & \mathcal{M} \arrow[dl, "p"] \\
& \Delta[1] &
\end{tikzcd}$$
commutes

(ii) $s|_{\mathcal{D} \times \{1\}} = h_1$

(iii) $s|_{\mathcal{D} \times \{0\}} = h_0 \circ g$

(iv) $s|_{\{d\} \times \Delta[1]}$ is p-Cartesian $\forall d \in \mathcal{D}$.

Rem The definition says that g "associated" to $0 \xrightarrow{f} 1$ in base $\Delta[1]$ satisfies the relation like f^* in classical situation of Grothendieck fibrations, namely if the equivalences h_0 and h_1 are identity, then $s|_{\{y\}\times\Delta[1]}$ is a p-Cartesian morphism $g(y) \xrightarrow{\tilde{f}} y$.

Prop 5.2.1.4 ((Construction of g from $p: \mathcal{M} \xrightarrow{\text{Cart. fib.}} \Delta[1]$)

Let $p: \mathcal{M} \to \Delta[1]$ be a Cartesian fibration and let $h_0: \mathcal{C} \to p^{-1}(0)$ and $h_1: \mathcal{D} \to p^{-1}(1)$ be equivs of quasicategories. Then there exists an associated functor $g: \mathcal{D} \to \mathcal{C}$.

Pr

$$\begin{tikzcd}
\mathcal{D}^\flat \times \{1\} \arrow[r, "h_1"] \arrow[d, "\text{marked anodyne}"'] & \mathcal{M}^\natural \arrow[d, "p"] \\
\mathcal{D}^\flat \times (\Delta[1])^\# \arrow[r, "pr_2"'] \arrow[ur, dashed, "\exists r"] & (\Delta[1])^\#
\end{tikzcd}$$

p Cartesian fib

because of form $id_{\mathcal{D}^\flat} \times (\text{type (ii) } n=1)$

$\Longrightarrow$ $r|_{\mathcal{D}\times\{1\}} = h_1$, and $r|_{\{d\}\times\Delta[1]}$ is p-Cartesian, as we want, but to get the other conditions needed on s in definition of associated functor, we need to alter r on $\mathcal{D}\times\{0\}$.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "r_0"'] \arrow[rr, dashed, bend left, "\text{def. } g"] & p^{-1}(0) \arrow[r, "\simeq", "h_0^{-1}"'] \arrow[rr, bend right=40, "\mathrm{id}_{p^{-1}(0)}"'] & \mathcal{C} \arrow[r, "\simeq", "h_0"'] \arrow[d, phantom, "\Downarrow \text{nat. equiv}"] & p^{-1}(0) \\
& & {} &
\end{tikzcd}$$

r_0 $\|$ $r|_{\mathcal{D}\times\{0\}}$

$\Longrightarrow$ $h_0 \circ g$ is naturally equivalent to r_0.

Let $e : \mathcal{D}\times\Delta[1] \longrightarrow \mathcal{M}$ be a nat. equiv $h_0\circ g \Rightarrow r_0$.

both level-wise p-Cartesian

$$\begin{tikzcd}
h_0\circ g \arrow[r, Rightarrow, "e"] & r_0 \arrow[r, Rightarrow, "r"] & r_1
\end{tikzcd}$$

is $r' : \mathcal{D}\times\Lambda^1[2] \longrightarrow \mathcal{M}$

$$\begin{tikzcd}
\mathcal{D}^\flat \times (\Lambda^1[2])^\# \arrow[r, "r'"] \arrow[d, hook, "\substack{\text{marked}\\ \text{anodyne}}"'] & \mathcal{M}^\natural \arrow[d, "p"', "\text{Cartesian fib}"] \\
\mathcal{D}^\flat \times (\Delta[2])^\# \arrow[r, "\substack{pr_2 \text{ composed w/}\\ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}"'] \arrow[ur, dashed, "\exists\, \bar{s}"] & (\Delta[1])^\#
\end{tikzcd}$$

see Appendix

Let $s := \bar{s}|_{\mathcal{D}\times\Delta^{\{0,2\}}} : \mathcal{D}\times\Delta[1] \longrightarrow \mathcal{M}$.

Then s satisfies the conditions needed to show g is associated to p, h_0, h_1.

□

Rem 5.2.1.6 There is a bijective correspondence between equivalence classes of functors $\mathcal{D} \to \mathcal{C}$ and equivalence classes of Cartesian fibrations $p : \mathcal{M} \to \Delta[1]$ equipped with equivalences $\mathcal{C} \to p^{-1}\{0\}$ and $\mathcal{D} \to p^{-1}\{1\}$.

Rem A similar statement holds for functors $\mathcal{C} \to \mathcal{D}$ and coCartesian fibrations $p: \mathcal{M} \to \Delta[1]$ equipped with equivalences $\mathcal{C} \to p^{-1}\{0\}$ and $\mathcal{D} \to p^{-1}\{1\}$.

Lurie's Definition of Adjunction

Def 5.2.2.1 (Lurie) Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories. An adjunction between $\mathcal{C}$ and $\mathcal{D}$ is a functor $q: \mathcal{M} \to \Delta[1]$ that is both a Cartesian fibration and a coCartesian fibration, together with equivalences $\mathcal{C} \to q^{-1}(0)$ and $\mathcal{D} \to q^{-1}(1)$.

Def In this situation, if g is associated to p as a Cartesian fibration, and f is associated to p as a coCartesian fibration, we say g is right adjoint to f and f is left adjoint to g.

Ex (Cor 5.2.4.5) Let $\mathcal{C}$ and $\mathcal{D}$ be categories enriched in Kan complexes. Suppose

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is an enriched adjunction, i.e., we have a binatural isomorphism of Kan complexes

$$\mathcal{D}(Fc, d) \cong \mathcal{C}(c, Gd).$$

Then $N^{simp}(F)$ and $N^{simp}(G)$ are adjoint functors between quasicategories.

Ex (Prop 5.2.4.6) Let $F \dashv G$ be a simplicial Quillen adjunction between simplicial model categories. Let $\mathcal{M}$ be the simplicial category corresponding to the adjunction $F \dashv G$, i.e. $\mathcal{C}, \mathcal{D} \underset{full}{\subseteq} \mathcal{M}$ and $\mathcal{M}(c,d) = \mathcal{C}(c, Gd) \cong \mathcal{D}(Fc, d)$.
Then $N^{simp}(\mathcal{M}^{o}) \longrightarrow \Delta[0]$ is an adj between $N^{simp}(\mathcal{C}^{o})$ and $N^{simp}(\mathcal{D}^{o})$.
See also Rem 5.2.4.7.

Left Adjoints Preserve Colimits

To prove left adjoints preserve colimits, Lurie needs a few lemmas and propositions.

Prop Let $p: K \longrightarrow \mathcal{C}$ be a diagram in a quasicategory $\mathcal{C}$ of shape K. Let $\bar{p}: K * 1 \longrightarrow \mathcal{C}$ be a cocone.
Then $\bar{p}$ is a colimiting cocone iff $\mathcal{C}_{\bar{p}/} \longrightarrow \mathcal{C}_{p/}$ is a trivial fibration.

Prop 2.1.2.5 (Compare Thm 3.19 of Joyal)

Consider $A \xhookrightarrow[\text{mono}]{i} B \xrightarrow{p} S \xrightarrow[\substack{\text{inner}\\ \text{fib}}]{\pi} T$.

If i is right anodyne, or if π is a left fib, then

$$S_{p/} \longrightarrow S_{p\circ i/} \underset{T_{\pi\circ p\circ i/}}{\times} T_{\pi\circ p/}$$

is a trivial fibration.

Lemma 5.2.3.1 Suppose

$$\begin{tikzcd} K\times\Delta[1] \arrow[rr, "P"] \arrow[dr] & & \mathcal{M} \arrow[dl, "q"] \\ & \Delta[1] & \end{tikzcd}$$

commutes in SSet, $\mathcal{M}$ is a quasicategory, and $P|_{\{k\}\times\Delta[1]}$ is q-coCartesian for every vertex k of K. Let $p := P|_{K\times\{0\}}$. Then the induced map

$$\begin{tikzcd} \mathcal{M}_{P/} \arrow[rr, "v"] \arrow[d] & & \mathcal{M}_{p/} \arrow[d] \\ \mathcal{M} \arrow[dr] & & \mathcal{M} \arrow[dl] \\ & \Delta[1] & \end{tikzcd}$$

fibers over 1 to a trivial fibration, i.e.

$$v_1 : \mathcal{M}_{P/} \times_{\Delta[1]} \{1\} \longrightarrow \mathcal{M}_{p/} \times_{\Delta[1]} \{1\}$$

is a trivial fibration.

Rem by Georgios

This is like the universality of unit arrows or counit arrows in classical descriptions of adjunctions.

Lemma 5.2.3.4 Let $q: \mathcal{M} \to \Delta[1]$ be a Cartesian fibration, and $\mathcal{C} := q^{-1}(0)$. Then the inclusion $\mathcal{C} \subseteq \mathcal{M}$ preserves all colimits which exist in $\mathcal{C}$.

Recall {trivial fibrations} $\subseteq$ {categorical fibrations}
Every trivial fibration is a categorical fibration.

Prop 5.2.3.5 Suppose $f: \mathcal{C} \to \mathcal{D}$ is a left adjoint between quasicategories. Then f preserves all colimits which exist in $\mathcal{C}$.

Pr Let $p: K \to \mathcal{C}$ be a K-shaped diagram in $\mathcal{C}$, and $\bar{p}: K * 1 \to \mathcal{C}$ a colimiting cocone of p. We want to show that $f \circ \bar{p}$ is a colimiting cocone of $f \circ p$.

Let $q: \mathcal{M} \to \Delta[1]$ be an adjunction with $\mathcal{C} = q^{-1}(0)$ and $\mathcal{D} = q^{-1}(1)$, and f is associated (use Prop 5.2.1.3 (1)).

Recall $f \circ \bar{p}$ is a colimiting cocone for $f \circ p$

if and only if $\mathcal{D}_{f\circ\bar{p}/} \xrightarrow{\text{def: } \varphi_1} \mathcal{D}_{f\circ p/}$ is a trivial fibration, so we show φ_1 is a trivial fibration.

Claim φ_1 is a categorical equivalence.

Pr f is associated to q, so $\exists\ s : \mathcal{C}\times\Delta[1] \to \mathcal{M}$ such that (i)

$$\begin{tikzcd} \mathcal{C}\times\Delta[1] \arrow[r, "s"] \arrow[dr, "pr_2"'] & \mathcal{M} \arrow[d, "q"] \\ & \Delta[1] \end{tikzcd}$$

commutes

(ii) $s|_{\mathcal{C}\times\{0\}} = id_{\mathcal{C}}$

(iii) $s|_{\mathcal{C}\times\{1\}} = f$

(iv) $s|_{\{c\}\times\Delta[1]}$ is q-coCartesian $\forall c\in\mathcal{C}$.

Have

$$\begin{tikzcd} & & \mathcal{C}\times\Delta[1] \arrow[dr, "s"] & & \\ K\times\{1\} \arrow[r, hook, "\text{right anodyne}"'] \arrow[rrr, bend right, "= f\circ p \text{ by (iii)}"'] & K\times\Delta[1] \arrow[ur, "p\times id"] \arrow[rr, "\text{def: } P"'] & & \mathcal{M} \arrow[r, "q"'] & \Delta[1] \end{tikzcd}$$

Prop 2.1.2.5 $\Longrightarrow$ $\mathcal{M}_{P/} \xrightarrow{\text{def: } u} \mathcal{M}_{f\circ p/} \times_{\Delta[1]_{q\circ f\circ p/}} \Delta[1]_{q\circ P/}$

is a trival fibration.

But both $\Delta[1]_{q\circ f\circ p/}$ and $\Delta[1]_{q\circ P/}$ are the terminal simplicial set, so u is $\mathcal{M}_{P/} \xrightarrow[\text{trivial fib}]{u} \mathcal{M}_{f\circ p/}$.

$$\left(\Delta[1]_{q\circ f\circ p/}\right)_n = \left\{ \begin{tikzcd} & K \arrow[dl] \arrow[dr, "q\circ f\circ p = \mathrm{const}_1"] & \\ K * \Delta[n] \arrow[rr] & & \Delta[1] \end{tikzcd} \right\} = \{1\} \quad \checkmark$$

$$\left(\Delta[1]_{q\circ P/}\right)_n = \left\{ \begin{tikzcd} K \times \Delta[1] \arrow[d, hook] \arrow[dr, "q\circ P"] & \\ \underbrace{(K\times\Delta[1])}_{\substack{\text{end maps to } 1\text{, so } \Delta[n] \text{ can} \\ \text{only map to } 1}} * \Delta[n] \arrow[r] & \Delta[1] \end{tikzcd} \right\} = pt \quad \checkmark$$

We have a similar diagram and argument for $K^{\triangleright} := K * 1$ and $\bar{p}$ and $\bar{P}$:

$$\begin{tikzcd} & & \mathcal{E}\times\Delta[1] \arrow[dr, "s"] & & \\ K^{\triangleright}\times\{1\} \arrow[r, hook, "\text{right}"', "\text{anodyne}"'] \arrow[rrr, bend right, "f\circ\bar{p} \text{ by (iii)}"'] & K^{\triangleright}\times\Delta[1] \arrow[ur, "\bar{p}\times id"] \arrow[rr, "\text{def: } \bar{P}"'] & & \mathcal{M} \arrow[r, "q"'] & \Delta[1]. \end{tikzcd}$$

Prop 2.1.2.5 $\Longrightarrow$ $\mathcal{M}_{\bar{P}/} \xrightarrow{\text{def } \bar{u}} \mathcal{M}_{f\circ\bar{p}/}$ is a trivial fibration.

We also know that $K^{\triangleright} \xrightarrow{\bar{P}} \mathcal{E} \hookrightarrow \mathcal{M}$ is a colimiting cocone of $K \xrightarrow{P} \mathcal{E} \hookrightarrow \mathcal{M}$ because the inclusion $\mathcal{E} \hookrightarrow \mathcal{M}$ preserves colimits (Lemma 5.2.3.4).

So $\mathcal{M}_{\bar{p}/} \xrightarrow{\text{def: } \varphi'} \mathcal{M}_{p/}$ is a trivial fibration.

We have a commutative diagram in SSet.

$$\begin{tikzcd}
\mathcal{M}_{\bar{p}/} \arrow[r, "\varphi'", "\text{triv. fib.}"'] & \mathcal{M}_{p/} \\
\mathcal{M}_{\bar{p}/} \arrow[u, "\bar{v}"] \arrow[r, "\psi"] \arrow[d, "\bar{u}"', "\substack{\text{triv.}\\ \text{fib.}}"] & \mathcal{M}_{p/} \arrow[u, "v"'] \arrow[d, "u", "\substack{\text{triv.}\\ \text{fib.}}"'] \\
\mathcal{M}_{f\circ\bar{p}/} \arrow[r, "\varphi"'] & \mathcal{M}_{f\circ p/}
\end{tikzcd}$$

induced by precomposing P w/ $K\times\{0\} \hookrightarrow K\times\Delta[1]$

Notice this is actually a diagram <u>over $\Delta[1]$</u>, so we can fiber over 1, that is, apply $- \times_{\Delta[1]} \{1\}$ to the whole diagram.

Each trivial fibration fibers to a trivial fibration, for instance, for u:

$$\begin{tikzcd}
\text{Fiber} \arrow[r] \arrow[d, dotted, "u\times_{\Delta[1]}\{1\}"'] \arrow[dd, bend right=60] & \mathcal{M}_{p/} \arrow[d, "u", "\substack{\text{triv.}\\ \text{fib.}}"'] \\
\text{Fiber} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & \mathcal{M}_{f\circ p/} \arrow[d] \\
1 \arrow[r, hook] & \Delta[1]
\end{tikzcd}$$

outer square and bottom square pullbacks,
$\Rightarrow$ top square pullback
$\Rightarrow u\times_{\Delta[1]}\{1\}$ is a triv. fib. (as pullback of a triv. fib.)

The maps of fibers v_1 and $\bar{v}_1$ are trivial fibrations by Lemma 5.2.3.1 because each $s|_{\{c\}\times\Delta[1]}$ is a q-coCartesian edge of $\mathcal{M}$.

Recall that trivial fibrations are categorical equivalences, which have the 3-for-2 property. 3-for-2 applied to maps of fibers $\Rightarrow \psi_1$ is a categorical equivalence.

Again applying 3-for-2 on fibers, φ_1 is a categorical equivalence (notice, the fiber map of φ is the φ_1 we were originally interested in!).

End of proof of
claim that φ_1 is
a categorical equiv. ✓

<u>Claim</u> $\varphi_1 : \mathcal{D}_{fop/} \longrightarrow \mathcal{D}_{fop/}$ is a trivial fib.

<u>Pr</u> φ_1 is a left fibration, so it is an inner fib and any morphism $b \to \varphi_1(a)$ has a φ_1-preimage with codomain a. In particular, each equivalence $b \simeq \varphi_1(a)$ has such a pre-image. Since φ_1 is a categorical equivalence, φ_1 reflects equivalences,

So any preimage of $b \simeq \varphi_1(a)$ is an equivalence.

$\Rightarrow$ φ_1 is a categorical fibration

(and φ_1 is already known to be a categorical equivalence).

$\Rightarrow$ φ_1 is a trivial fibration ✓

Finally $f \circ \bar{p}$ is a colimiting cocone of $f \circ p$, and f preserves colimits. □

Equivalence of the Two Definitions of Adjunction

Joyal's definition and Lurie's definition are equivalent, but it is nontrivial to prove. For work on this see

Rem 4.4.5 of Riehl-Verity, 2-Cat. Theory of QCats

and

Prop 4.1.20 and Obs. 4.1.22 of Riehl-Verity, Fibrations and Yoneda's Lemma in an ∞-Cosmos.

See also

<u>Prop</u> 5.2.2.8 (Lurie HTT) Let $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$ be functors between quasicategories. T.F.A.E.

(i) f is left adjoint to g

(ii) There exists a natural transf $\eta : id_{\mathcal{C}} \to g \circ f$ such that

$$\mathbb{C}\mathcal{D}(fc,d) \xrightarrow{\mathbb{C}g} \mathbb{C}\mathcal{C}(gfc,gd) \xrightarrow{\circ\eta_c} \mathbb{C}\mathcal{C}(c,gd)$$

is a weak homotopy equivalence $\forall c \in \mathcal{C}, d \in \mathcal{D}$.

<u>Appendix on (Weakly) Saturated Classes of Morphisms in a 1-Category</u>

<u>Def</u> (Joyal D.2.2) A class $\mathcal{A}$ of maps in a cocomplete 1-category is called <u>saturated</u> if

- $\mathcal{A}$ contains the isomorphisms
- $\mathcal{A}$ is closed under composition
- $\mathcal{A}$ is closed under pushouts
- $\mathcal{A}$ is closed under retracts, and
- $\mathcal{A}$ is closed under transfinite composition.

<u>Rem</u> Lurie calls "Joyal saturated" instead "weakly saturated."

<u>Prop</u> Any saturated class is closed under coproducts. (Follows from Joyal's Prop D.2.1.)

Recall the saturated class of marked anodyne morphisms in Set^+_Δ on page 7.

<u>Claim</u> Let $\mathcal{D}$ be any quasicategory (or even simplicial set). Then

$$\mathcal{D}^\flat \times (\Lambda^1[2])^\sharp \hookrightarrow \mathcal{D}^\flat \times (\Delta[2])^\sharp$$

is marked anodyne.

<u>Pr</u> It suffices to show $(\Lambda^1[2])^\sharp \hookrightarrow (\Delta[2])^\sharp$ is marked anodyne, since we know (mono)x(marked anodyne) is marked anodyne.

$$\begin{tikzcd}
(\Lambda^1[2])^\flat \arrow[r] \arrow[d, "\substack{\text{marked}\\ \text{anodyne}\\ \text{type (i)}}"'] \arrow[dr, phantom, "\text{pushout}"] & (\Lambda^1[2])^\sharp \arrow[d, "\Rightarrow \substack{\text{marked}\\ \text{anodyne}}"] \arrow[dd, bend left=60] \\
(\Delta[2])^\flat \arrow[r] & (\Lambda^1[2])^\sharp \coprod_{(\Lambda^1[2])^\flat} (\Delta[2])^\flat \arrow[d, hook, "\substack{\text{marked}\\ \text{anodyne}\\ \text{type (iii)}}"'] \\
& (\Delta[2])^\sharp
\end{tikzcd}$$

$\Longrightarrow$ desired composite is marked anodyne

$\square$

Exercise

Let $\mathcal{M} \xrightarrow{p} [1]$ be a Grothendieck bifibration (i.e. both a Grothendieck fibration and Grothendieck opfibration) between classical 1-categories,

$[1] := \{0 < 1\}$.

Follow the classical construction on pages 9 and 10 to construct functors

$$\begin{tikzcd} p^{-1}(0) \arrow[r, bend left] & p^{-1}(1) \arrow[l, bend left] \end{tikzcd}$$

and prove they are adjoint functors in the classical sense of 1-category theory.

Conversely, given a classical adjunction of 1-categories, construct an associated Grothendieck bifibration with codomain $[1]$.