Cartesian Model Structure

Motivating Problem

Many constructions in category theory produce data that only characterizes objects up to unique isomorphism.

One can make choices of representatives but these choices will usually fail to be functorial.

Ex $f \in CAlg(A,B)$ we obtain functors

$$Mod(A) \xrightarrow{B\otimes_A -} Mod(B) \underset{d^1}{\overset{d^0}{\rightrightarrows}} Mod(B\otimes_A B)$$

d^0 is induced by $B \xrightarrow{B\otimes e} B\otimes_A B$

d^1 is induced by $B \xrightarrow{e\otimes_A B} B\otimes_A B$

Now $(B\otimes_A e)\circ f = (e\otimes_A B)\circ f =: c$

but $d^1 d^0$ and $d^1 \circ d^0$ are only canonically isomorphic to the functor induced by c.

Classically one gets past this lack of functoriality by either of the following equiv approaches

(a) Relaxing the notion of functoriality to saying we have a pseudofunctor $\Delta^+ \to Cat$ (so the unit and associativity conditions on a functor are relaxed to natural isos)

(b) Constructing a Groth. fib $\begin{matrix}\mathcal{E}\\ \downarrow p\\ (\Delta^+)^{op}\end{matrix}$

where $p^{-1}([N]) \simeq Mod(B^{\otimes_A N+1})$

and given $M \in p^{-1}([N])$ and a map $[N] \xrightarrow{\bar{f}} [k] \in \Delta^+$ we have

a map $M' \xrightarrow{f} M \in \mathcal{E}$ with $pf = \bar{f}$ and this f is determined up to a universal property

(If we chose another $f': M'' \to M$ with this property we obtain canonical equivalences $M'' \xrightarrow{\simeq} M'$ over the id$[k]$.)

Approach (b) avoids making choices (fixing structure) and instead emphasizes a property (such choices can be made). This makes this approach a bit easier to construct and a lot easier once we start trying to construct maps that respect our data.

The implication (b)$\Rightarrow$(a) (straightening) is a rigidification result, which justifies (b) is usually the result one is after.

(a)$\Rightarrow$(b) (unstraightening) (or recode) is essentially regarding a commutative diagram as a coherently commutative diagram.

In ∞-categories it is even less natural to make choices since most ∞-categories arise from a multitude of possible models and/or from various universal properties.

Recollections (from Kim's and Tom's talks)

Defn Let $p: X \to S$ be an inner fib of sSets. $\Delta^1 \xrightarrow{f} X$ (classifying $x \xrightarrow{f} y$) is **p-cart** if the induced map $X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$ is a trivial Kan fib.

Defn $p: X \to S$ map of sSets is a **cartesian fibration** if

1) p is an inner fib
2) $\forall\ \Delta^1 \xrightarrow{\bar{f}} S$ (classifying $\bar{x} \xrightarrow{\bar{f}} \bar{y}$) and every $y \in X$ s.t. $p(y) = \bar{y}$ $\exists$ p-cart edge $\Delta^1 \xrightarrow{f} S$ classifying $x \xrightarrow{f} y$ s.t. $p(f) = \bar{f}$.

Defn A **marked sSet** $(X, \mathcal{E})$ is a pair $X \in sSet$ and $\mathcal{E} \subseteq X_1$ (marked edges) s.t. $\mathcal{E} \supseteq s_0 X_0$. A map of such is a map of sSets sending marked edges to marked edges. Denote this cat by $sSet^+$. ($sSet^+$ is a loc pres. closed cartesian category gen by $\{(\Delta^n, \flat)\}_{n\geq 0}$, $s_0\Delta^n_0 \subseteq \mathcal{E} \subseteq \Delta^n$)

Construction The obvious forgetful functor $sSet^+ \longrightarrow sSet$ has a left adjoint

$sSet \xrightarrow{(\ )^\flat} sSet^+$ (mark only deg edges)

and a right adjoint

$sSet \xrightarrow{(\ )^\#} sSet^+$ (mark all edges)

Both these functors preserve colimits and are cartesian monoidal.

These make $sSet^+$ into a simplicially enriched (tensored + cotensored) category in two distinct ways:

$$sSet^{+,\flat}((X,\mathcal{E}),(Y,\mathcal{E}'))_n = sSet^+(\Delta^{n\flat}\times(X,\mathcal{E}),(Y,\mathcal{E}'))$$
$$\cup\!\!\!\mid$$
$$sSet^{\#}((X,\mathcal{E}),(Y,\mathcal{E}'))_n = sSet^+(\Delta^{n\#}\times(X,\mathcal{E}),(Y,\mathcal{E}'))$$

Const: Given a cartesian fib $p: X \to S$ Let $X^\natural = (X, \mathcal{E}) \in sSet^+$ where $\mathcal{E}$ is the subset of p-cartesian edges in X.

E.g.: If $p: X \to *$ is an inner fib, i.e. X is an ∞-cat then p is a Cartesian fib and the marked edges of $X^\natural$ are precisely the equivalences. (This holds more gen when $* \leftarrow S$ a Kan complex)

Not For $S \in sSet$, let $sSet^+_{/S} = sSet^+_{/S^\#}$.

Thm (with definitions to follow)

Let $S \in sSet$. Then $\exists$ a left proper combinatorial model structure on $sSet^+_{/S}$ whose

- Cofibrations are precisely the monos (= maps which are cofibs in sSet)
- weak equivalences are the cartesian equivalences

Moreover with respect to the tensoring w/ sSet via $sSet \xrightarrow{\times S} sSet_{/S} \xrightarrow{\#} sSet^+_{/S}$ and hom $sSet^{+,\#}_{/S}$, $sSet^+$ is a $sSet_{Kan\text{-}Quillen}$-model cat.

Similarly for $sSet \xrightarrow{\times S} sSet_{/S} \xrightarrow{(\ )^\flat} sSet^+_{/S}$ and $sSet^{+,\flat}_{/S}$ is a $sSet_{Joyal}$-model cat.

The fibrant objects of $sSet^+_{/S}$ are precisely the objects of the form $p: X^\natural \to S$ for some cartesian fib $p: X \to S$.

Recall $f: X \to Y \in sSet$ is a categorical equiv iff $\Leftrightarrow$ $\mathfrak{C}[X] \to \mathfrak{C}[Y]$ equiv of sCat

& $\forall$ ∞-cat Z (i.e. fibrant) $\tau_0 Y^Z \to \tau_0 X^Z$ is an equiv of cats

If X, Y ∞-cats f is a cat equiv if either of the equiv conditions hold

(1) $\forall K \in sSet$ $X^K \xrightarrow{f^K} Y^K$ is a cat equiv

(2) $\forall K \in sSet$ f^K induces a weak equivalence of maximal sub Kan complexes

Exer If $p: Y \to S$ cart fib then $Map^\flat_{/S}((X,\mathcal{E}), Y^\natural)$ is an ∞-cat and $Map^\#_{/S}((X,\mathcal{E}), Y^\natural)$ is the largest Kan subcomplex

Prop/Defn $S \in sSet$, $f \in sSet^+_{/S}(X,Y)$

f is a cart equiv if either of the following equiv conditions holds

1) $\forall Z \to S$ cart fib $Map^\flat_{/S}(Y, Z^\natural) \to Map^\flat_{/S}(X, Z^\natural)$ is a cat equiv
2) $\forall Z \to S$ cart fib $Map^\#_{/S}(Y, Z^\natural) \to Map^\#_{/S}(X, Z^\natural)$ is a homotopy equiv

Remark If $\begin{matrix}X & \xrightarrow{f} & Y\\ & \searrow\ \swarrow & \\ & S & \end{matrix}$ is a map of cart fibs over S

TFAE

1) f is a cart equiv
2) f has a strong homotopy inverse g i.e. $id_X \amalg g\circ f \hookrightarrow Map^\flat_{/S}(X, X)$ extends over a contractible Kan complex and similarly for $f\circ g$
3) f induces categorical equivalences fiberwise.

recall cart fib $\Rightarrow$ inner $\Downarrow$ fiberwise ∞-cats

Let
- $e^0 = sSet_{/S}$ w/ Joyal model str over (cofibs = monos, w.e. = cat equiv)
- $e^1 = sSet^+_{/S}$ w/ marked model str
- $e^2 = e^1$ w/ w.e. those maps $\begin{matrix}X \to Y\\ \searrow\ \swarrow\\ S\end{matrix}$ s.t. $X^\# \to Y^\#$ is a cartesian equiv
- $e^3 = sSet_{/S}$ w/ contravariant model str $\begin{matrix}X \to Y\\ \searrow\ \swarrow\\ S\end{matrix}$, $X^\triangleright \amalg_X S \to Y^\triangleright \amalg_Y S$ is a cat equiv
- $e_4 = sSet_{/S}$ w/ Kan-Quillen model str

Thm There is a chain of Quillen left adjoint

$$e_0 \xrightarrow[\text{Q equiv when } S \text{ is Kan}]{(\ )^\flat} e_1 \xrightarrow{id} e_2 \xrightarrow[\text{equiv}]{\text{forgetful}} e_3 \xrightarrow[\text{Q equiv when } S \text{ is Kan}]{id} e_4$$

Straightening

from last term $X \xrightarrow{f} S$

set $St_S(X) \in sCat(\mathfrak{C}[S]^{op}, sSet_{KQ})$

By $D \in \mathfrak{C}[S]^{op} \mapsto \mathfrak{C}[X^\triangleright \amalg_X S](D, *)$ $= St_S(X)(D)$

Note $g: C \to D \in \mathfrak{C}[S]$ induces $\mathfrak{C}[X^\triangleright \amalg_X S](D,*) \xrightarrow{g^*} \mathfrak{C}[X^\triangleright \amalg_X S](C,*)$

For $(X,\mathcal{E}) \in sSet^+_{/S}$ we obtain

$St_S(X) \in sCat(\mathfrak{C}[S]^{op}, sSet^+_{marked})$ (# maps)

simplicial functor is as before we just need to specify which edges to mark in $St_S(X)(C)$ for $C \in \mathfrak{C}[S]^{op}$

give $\bar{g} \in \mathcal{E} \subseteq X$, obtain $\Delta^1 \xrightarrow{g} \ldots$ $a \to b$

$\Delta^2 = \Delta^{1\triangleright} \longrightarrow X^\triangleright$ and $\mathfrak{C}[\Delta^2] \to \mathfrak{C}[X^\triangleright \amalg_X S]$

$a \xrightarrow{g} b$, $\bar{a} \searrow\ \swarrow \bar{b}$ $*$ $\rightsquigarrow$ defines 1-simplex from $\bar{a}$ to $g^*\bar{b}$ in $St_S(X)(a)$

We mark these edges and also mark any edge obtained by p.b. along such morphism of $\mathfrak{C}[S](C,D)$.

Thm $sSet^+_{/S} \xrightarrow{St} sCat(\mathfrak{C}[S]^{op}, sSet^+)$ is a Quillen equiv.