LIE ALGEBRAS: EXERCISES

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This list of problems will grow over the term. It will generally include problems related to future material, so do not worry if you do not know how to approach, or even understand the statement of, the later problems.

(1) Classify all 2 dimensional Lie algebras up to isomorphism. (Hint: there are only two, use a change of basis to put them into a standard form).
(2) Write done the multiplication table for sl_3. Put all of the terms with diagonal entries at an extreme end of the table.
(3) A quadratic form on V is a homogeneous degree 2 polynomial on a finite dimensional vector space V over a field k. Show that every symmetric bilinear form defines a quadratic form and that if $2 \neq 0 \in k$ then every quadratic form determines a symmetric bilinear form.
(4) Let $\dim V = 2m$ and consider the subset $o(V) = o(2m,k)$ of $gl(V)$ whose elements are linear transformations x such that $(xv,w) = -(v,xw)$ for all $v,w \in V$ for the bilinear form defined by:
$$s = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$
Show that this a Lie subalgebra of gl_V and determine a basis. What is the dimension of $o(V)$?
(5) Show that if $x \in t_n$ (upper triangular matrices) and $y \in d_n$ (diagonal matrices) then $[x,y] \in n_n$ (the strictly upper triangular matrices).
(6) Show that $[t_n, t_n] = n_n$.
(7) Show that the assignment $x \mapsto \mathrm{ad}x$ defines a map of Lie algebras $\mathrm{ad}l \to Der(l)$.
(8) What is $\ker \mathrm{ad}$?
(9) Show that any simple algebra is a subalgebra of $gl(V)$ for some V. (Consider the adjoint representation).
(10) Show that $A_1 \cong B_1 \cong C_1$, D_1 is abelian, $B_2 \cong C_2$ and $D_3 \cong A_3$.
(11) (related to previous exercise) What can you say about D_2?
(12) Show that for each classical Lie algebra L, $[L,L] = L$.
(13) Show that L is solvable if and only if there exists a chain of subalgebras
$$L = L_0 > L_1 > \cdots > L_n = 0$$
such that L_{i+1} is an ideal of L_i and L_i/L_{i+1} is abelian.
(14) Show that L is solvable (resp. nilpotent) if and only if the image $\mathrm{ad}L < gl(L)$ is solvable (resp. nilpotent).
(15) Let L be nilpotent and K a proper subalgebra of L then show K is a proper subalgebra of $N_L K$.
(16) Show that every nilpotent Lie algebra has an ideal of codimension 1.

(17) Show that two commuting semisimple endomorphisms can be simultaneously diagonalized.
(18) Let x, y be two commuting matrices. Show that $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y_n$. Show that this can fail if x and y do not commute.
(19) Prove that if L is nilpotent, the Killing form of L is identically zero. Does the converse hold?
(20) Prove that L is solvable if and only if $[L,L]$ is in the radical of the Killing form.
(21) Show that the three dimensional vector space with basis x,y,z and brackets: $[x,y]=z$, $[x,z]=y$, $[y,z]=0$ is a Lie algebra M.
(22) Calculate the matrix of the Killing form of M and determine its radical.
(23) Is M simple? nilpotent? solvable?
(24) Compute the dual basis, relative to the Killing form, of the standard basis of sl_2.
(25) Let $L = \bigoplus L_i$ be the decomposition of a semisimple Lie algebra into simple ideals. Show that the semisimple and nilpotent parts of $x \in L$ are the sums of the semisimple and nilpotent parts in the L_i respectively.
(26) Using the standard basis for sl_2 write down the Casimir element corresponding to the adjoint representation.
(27) Using the standard basis for sl_3 write down the Casimir element corresponding to the standard three dimensional representation.
(28) Show that if L is solvable then every irreducible representation of L is one dimensional.
(29) A Lie algebra is reductive if Rad(L)=$Z(L)$. Show
  (i) If L is reductive then L is a completely reducible ad$L$-module (use Weyl's theorem in the case ad$L \neq 0$). In particular L is the direct sum of $Z(L)$ and $[L,L]$ and the latter is either 0 or semisimple.
  (ii) The converse of the previous statement.
  (iii) If L is reductive then every finite dimensional representation of L such that $Z(L)$ is represented by semisimple endomorphisms is completely reducible.
(30) Use Lie algebra cohomology to classify the extensions of a 2-dimensional abelian Lie algebra by a 1-dimensional Lie algebra.
(31) Show that n_3 realizes a non-trivial extension.
(32) (If you know about the Lyndon-Hochschild-Serre spectral sequence) Calculate the cohomology of n_3 (to determine the differential use your knowledge of H_1).
(33) Consider $sl_2 < sl_3$ as a subalgebra via the inclusion in the top left corner. This makes sl_3 into an sl_2-representation. Decompose it into irreducibles.
(34) Let L be a simple Lie algebra. Let β, γ be two symmetric non-degenerate associative bilinear forms on L. Show that $\beta = \lambda\gamma$ for some non-zero scalar λ. (Hint: β induces an iso $L \to L^*$ and γ induces an iso $L^* \to L$, show that the composite is an isomorphism of L-modules and apply Schur's lemma.)
(35) Assume the following: sl_n is simple. Let κ denote the Killing form on sl_n. Show that $\kappa(x,y) = 2n\,\mathrm{tr}(xy)$.
(36) For sl_n calculate the root strings and Cartan integers.
(37) Prove that every 3-dimensional semisimple Lie algebra has the same root system as sl_2 and is therefore isomorphic to sl_2.

(38) Given a root α define the dual root $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$. Show that the set $\Phi^\vee$ of dual roots is again a root system with isomorphic Weyl group. Moreover show that $[\alpha^\vee, \beta^\vee) = [\alpha, \beta)$.

(39) Draw a picture of $\Phi^\vee$ in the cases of A_1, A_2, B_2, and G_2.

(40) Prove that the Weyl groups of $A_1 \times A_1, A_2, B_2$, and G_2 are dihedral of respective orders: 4, 6, 8, and 12. If Φ is any root system of rank 2, prove that its Weyl group must be one of these.