LIE THEORY: PROBLEM SET 1

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(1) Suggested reading [Lee03, Ch. I–3], [Kir08, Ch. 1–2], and potentially [Hsi00, Ch. 1–2]. (You are not expected to read all of these, but you might find it helpful.)

(2) Show that if $H < G$ is an open subgroup of a topological group G, then it is a closed subgroup.

(3) Suppose that $F: M \to N$ is a smooth map between two smooth manifolds and $p \in M$. Show that the pushforward map

$$F_*: T_pM \to T_{F(p)}N$$

satisfies

(i) F_* is linear at any point of M.

(ii) At any point of M, Id_{M*} is the identity matrix.

(iii) For any composable sequence of smooth maps

$$M \xrightarrow{F} N \xrightarrow{G} P$$

and a point p of M we have an equality

$$G_* \circ F_* = (G \circ F)_*.$$

(iv) If F is a diffeomorphism, then for any point of M, F_* is a linear isomorphism.

(4) Let $F: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$F(x,y) = (x^2 + 2xy + 3y, \cos x + \sin y + e^y, xy + 2x + 3).$$

Using the identity charts on the source and target and the standard bases $(\partial/\partial x, \partial/\partial y)$ and $(\partial/\partial u, \partial/\partial v, \partial/\partial w)$ calculate

(i) F_* at $(0,0)$.

(ii) F_* at $(1,2)$.

(iii) F_* at (p_1, p_2). What is the name of this matrix?

(5) The goal of this exercise is to show $\det_* B$ at A is $\det(A)tr(A^{-1}B)$, where $A \in GL_n(\mathbb{K})$ and $B \in M_n(\mathbb{K})$.

(i) Show that $\sum_{i,j} A_{i,j}B_{i,j} = tr(A^tB)$.

(ii) Show that $\frac{\partial}{\partial A_{i,j}} \det(A) = C_{i,j}$ where $C_{i,j}$ is the i,jth cofactor matrix obtained by taking the determinant of the matrix where we have deleted the ith row and jth column from A and multiplying by -1^{i+j}. The cofactors fit into a matrix C.

(iii) Use the previous calculation to determine $\det_*(B)$ at A in terms of the cofactor matrix.

(iv) Use (5.i) and Cramer's rule: $A^{-1}\det(A) = C^t$ to show that $\det_*(B)$ at A is $\det(A)tr(A^{-1}B)$.

References

[Hsi00] W.-Y. Hsiang, *Lectures on Lie groups*, Series on University Mathematics, vol. 2, World Scientific Publishing Co. Inc., River Edge, NJ, 2000. MR 1788014 (2002d:22001)

[Kir08] Alexander Kirillov, Jr., *An introduction to Lie groups and Lie algebras*, Cambridge Studies in Advanced Mathematics, vol. 113, Cambridge University Press, Cambridge, 2008. MR 2440737 (2009f:22001)

[Lee03] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003. MR 1930091 (2003k:58001)