LIE THEORY: PROBLEM SET 2

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1. Suggested reading the relevant parts of [Lee03, Ch. 3,5,12-14] .
2. Suppose that $F\colon M \to N$ is a smooth map between smooth manifolds and that M is path-connected. Show that if for all $p \in M$, $F_*\colon T_pM \to T_{F(p)}N$ is 0 then F is constant.
3. Let G be a Lie group and $H \subset G$ a subgroup and an embedded submanifold. Then show that H is a closed Lie subgroup by completing the following exercises:
   3.1. Shot that the restrictions of the multiplication and inversion maps are smooth.
   3.2. Show that H is closed. Hints:
      (i) Since H is locally euclidean to see it is closed it suffices to show that the limit point $g \in G$ of an arbitary sequence $\{h_i\} \in H$ lies in H. Let $U, W \subset G$ be open neighborhoods of the identity such that $\overline{W} \subset U$ and $H \cap U$ is closed in U. Show that there is a neighborhood $V \subset G$ of the identity such that $v_1^{-1}v_2 \in W$ for all $v_1, v_2 \in V$.
      (ii) Now $g^{-1}h_i$ is a sequence limiting to e. Taking i large we can assume $g^{-1}h_i$ is in V. Show that this implies $h_i^{-1}h_j$ is in W.
      (iii) Now uses that $H \cap U$ is closed in U to see that $g^{-1}h_j \in H$ and hence $g \in H$.
4. Compute the Lie bracket structure on the Lie algebra $\mathfrak{m}_(\mathbb{R})$ associated to $M_n(\mathbb{R})$.
5. Let U be an open neighborhood of the identity in a connected topological group G. Show that U generates (as a group) G. (Hint: Use an exercise from the last problem set).
6. Show every 1 dimensional Lie algebra is abelian, i.e., $[X, X] = 0 \forall X \in \mathfrak{g}$.
7. Show every 2 dimensional Lie algebra is either abelian or isomorphic to the following Lie algebra $\mathfrak{g}$ generated by X, Y such that $[X, Y] = Y$.

References

[Lee03] John M. Lee, *Introduction to smooth manifolds*, Graduate Texts in Mathematics, vol. 218, Springer-Verlag, New York, 2003. MR 1930091 (2003k:58001)