LIE THEORY: PROBLEM SET 5

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1. Suggested reading the relevant parts of [Lee11, Ch. 11,12].
2.
   2.1. Show that there exists a Lie group homomorphism $p: U(1) \to U(n)$ such that $\det \circ p = \mathrm{Id}$.
   2.2. Show that $U(n)$ is diffeomorphic to $U(1)xSU(n)$.
   2.3. Show that $U(n)$ and $U(1)xSU(n)$ are not isomorphic Lie groups when $n > 1$. [Hint: Isomorphic groups have isomorphic centers.]
3. Show that the quaternions $\mathbb{H} = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus \mathbb{R}$ is an associative but not commutative $\mathbb{R}$-algebra. If you are not familiar with the quaternions, the multiplicative relations are

$$ij = -ji \tag{0.1}$$

$$jk = -kj \tag{0.2}$$

$$ik = -ki \tag{0.3}$$

$$i^2 = j^2 = k^2 = -1 \tag{0.4}$$

4. For $x = a + bi + cj + dk \in \mathbb{H}$ let $x^* = a - bi - cj - dk$. Show that $(xy)^* = y^* x^*$ and that
$$\langle x, y \rangle = \frac{p^* q + q^* p}{2}$$
defines an inner product on $\mathbb{H}$.
5. Show that if $x \in \mathbb{H} \setminus 0$, then $x^{-1} = \frac{x^*}{\langle x, x \rangle}$.
6. Show that the unit quaternions (under this inner product) form a Lie group diffeomorphic to S^3.
7. Show that $SU(2)$ is isomorphic to the group S of unit quaternions and diffeomorphic to S^3.
8. Identify the Lie algebra on $Sp(n)$

REFERENCES

[Lee11] John M. Lee, *Introduction to topological manifolds*, second ed., Graduate Texts in Mathematics, vol. 202, Springer, New York, 2011. MR 2766102 (2011i:57001)