FOR COMPLEX ORIENTATIONS PRESERVING POWER OPERATIONS, p-TYPICALITY IS ATYPICAL

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Abstract. We show, for primes $p \leq 13$, that a number of well-known $MU_{(p)}$-rings do not admit the structure of commutative $MU_{(p)}$-algebras. These spectra have complex orientations that factor through the Brown-Peterson spectrum and correspond to p-typical formal group laws. We provide computations showing that such a factorization is incompatible with the power operations on complex cobordism. This implies, for example, that if E is a Landweber exact $MU_{(p)}$-ring whose associated formal group law is p-typical of positive height, then the canonical map $MU_{(p)} \to E$ is not a map of H_∞ ring spectra. It immediately follows that the standard p-typical orientations on BP, $E(n)$, and E_n do not rigidify to maps of E_∞ ring spectra. We conjecture that similar results hold for all primes.

1. Introduction

This paper arose out of the authors' attempts to address the long-standing open conjecture:

Conjecture 1.1. *For every prime p, the p-local Brown-Peterson spectrum BP admits an E_∞ ring structure.*

Quillen showed that the algebraic map $r_*\colon MU_{(p)*} \to BP_*$ classifying a universal p-typical formal group law over BP_*, could be realized topologically as the retraction map in a splitting of p-local complex cobordism:

$$BP \xrightarrow{s} MU_{(p)} \xrightarrow{r} BP. \tag{1.2}$$

This splitting plays a key role in many computational applications, especially in the Adams-Novikov spectral sequence.

One might hope that an E_∞ structure on BP could be exploited in a number of computations, in particular to prove the existence of differentials in the Adams-Novikov spectral sequence. In the case of MU, a very nice example of such a technique can be found in the recent work of Hopkins, Hill, and Ravenel on the Kervaire invariant one problem [HHR09]. In their proof, the E_∞ structure on MU plays a crucial role in demonstrating that certain elements in the Adams and Adams-Novikov spectral sequences must support differentials.

A number of attempts have been made to prove the above conjecture and there have been some positive results in this direction. Recently, Birgit Richter has shown that BP is at least $2(p^2+p-1)$ homotopy commutative [Ric06]. In unpublished work, Basterra and Mandell were able to show that BP admits an E_4-ring structure. There are also a number of results demonstrating that BP and its relatives admit various multiplicative structures compatible with those of MU [EKMM97, Str99, Goe01, Laz03].

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Since Quillen's splitting plays an important role in many BP computations, it is natural to ask whether either map in Equation 1.2 can be made into a map of E_∞ ring spectra. This splitting was shown to be A_∞ in [Laz04]. In this paper we consider the retraction map

$$r\colon MU_{(p)} \to BP$$

and maps of spectra factoring through r. The section

$$s\colon BP \to MU_{(p)}$$

has already been considered by Hu-Kriz-May; they have shown that there are no E_∞ ring maps whatsoever from BP into $MU_{(p)}$ [HKM01, 2.11] [BM04, App. B]. In fact, their proof yields the stronger result that there are no H_∞ ring maps from BP into $MU_{(p)}$.

An H_∞ ring spectrum can be thought of as an E_∞ ring spectrum up to homotopy; such spectra correspond to cohomology theories with a well-behaved theory of power operations in degree 0. To obtain power operations in other cohomological degrees, one needs the richer structure known as H^d_∞. The H^2_∞ structure on MU plays a prominent role in this paper. This structure arises from the E_∞ structure on BU via the "Thomification" functor [May77, IV.2]. The resulting power operations agree with the Steenrod operations in cobordism constructed in [tD68].

The central work of this paper is to compute the action of these power operations on $MU^{2*}_{(p)}$, modulo the kernel of r_*. These calculations yield obstructions to lifting a ring map

$$MU_{(p)} \to BP \to E$$

to a map of H_∞ ring spectra.

Theorem 1.3. *Suppose $f\colon MU_{(p)} \to E$ is map of H_∞ ring spectra satisfying:*

(1) *f factors through Quillen's map to BP.*
(2) *f induces a Landweber exact MU_*-module structure on E_*.*
(3) **Small Prime Condition:** *$p \in \{2, 3, 5, 7, 11, 13\}$.*

then π_*E is a $\mathbb{Q}$-algebra.

The Landweber exactness requirement is primarily a matter of convenience for the statement and proof of this result. Our computations have similar consequences for a more general class of p-typical spectra and the proof of Theorem 1.3—given at the end of Section 2—illustrates how one might apply the calculations in general. Details of our computational methods, as well as a more complete list of calculations, are given in Section 6.

As special cases of Theorem 1.3 we obtain the following:

Theorem 1.4. *Suppose the Small Prime Condition holds and $n \geq 1$. The standard p-typical orientations on E_n, $E(n)$, $BP\langle n\rangle$, and BP do not respect power operations. In particular, the corresponding MU-ring structures do not rigidify to commutative MU-algebra structures.*

Remark 1.5. The appearance of the Small Prime Condition in the above two theorems arises from limitations of our computational resources and the efficiency of our algorithms. There is no theoretical bound on the primes for which our methods apply.

Conjecture 1.6. *Theorems 1.3 and 1.4 hold without the Small Prime Condition.*

In closely related work, Matt Ando [And95] has constructed H_∞ maps from MU to E_n. Since these maps satisfy the second condition of Theorem 1.3 we have the following:

Corollary 1.7. *For the primes listed above, none of the H_∞ orientations on E_n constructed in* [And95] *are p-typical.*

Notation 1.8. Throughout this paper we will refer to a map of ring spectra $MU \to E$ as a (complex) orientation on E or an MU-ring structure. For convenience, we will henceforth assume all spectra are localized at a prime p. We will also use the shorthand

$$E^* \equiv E^*(*) = \pi_{-*}E$$

for the E cohomology of a point. We will use cohomological gradings throughout this paper.

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2. Main Theorems

Our work studies power operations arising from an H_∞ orientation. This naturally produces power operations in degree 0, and our first step is to observe that such an orientation under MU defines a wider family of power operations acting on even degrees.

Theorem 2.1 (See 3.13)**.** *Suppose $f\colon MU \to E$ is a map of H_∞ ring spectra, then for each $n \in \mathbb{Z}$ there is a map*

$$P_{C_p,E}\colon E^{2n} \to E^{2np}(BC_p)$$

making the following diagram commute.

$$\begin{array}{ccc} MU^{2n} & \xrightarrow{P_{C_p,MU}} & MU^{2pn}(BC_p) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ E^{2n} & \xrightarrow{P_{C_p,E}} & E^{2pn}(BC_p) \end{array}$$

Figure 2.1. Even-degree power operations induced by H_∞ orientation under MU.

These power operations are precisely tom Dieck's Steenrod operations in cobordism [tD68]; they form what is known as an H_∞^2 ring structure [BMMS86]. In Section 3.2 we provide an alternate construction of these operations, using the Thom isomorphism and the standard H_∞ structure on MU. In Theorem 3.13 we show that an orientation $MU \to E$ is H_∞ if and only if it is H_∞^2, and Theorem 2.1 follows from this.

Our applications rely on a simple observation following from Theorem 2.1: if one can find some $x \in MU^{2n}$ such that $f_*(x) = 0$, yet $f_*P_{C_p,MU}(x) \neq 0$, then the above square can not commute and therefore f can not be a map of H_∞ ring spectra. The difficulty in this approach is showing $f_*P_{C_p,MU}(x) \neq 0$. For well chosen x and f this is a strictly algebraic problem, although not a simple one.

2.1. **Reducing to an algebraic condition.** The theory of formal group laws provides a description of the ring $E^*(BC_p)$, which we describe below. The ring MU^* carries the universal formal group law, and so an orientation $f\colon MU \to E$ induces a formal group law on E. For any formal group law F, the p-series $[p]_F\xi$ and the reduced p-series $\langle p\rangle_F\,\xi$ are defined by the following equations (when clear from context, we will drop the subscript F):

$$\overbrace{\xi +_F \cdots +_F \xi}^{p \text{ times}} = [p]_F\,\xi = \xi \cdot \langle p\rangle_F\,\xi$$

The ring $E^*(BC_p)$ is isomorphic to the quotient ring $E^*[\![\xi]\!]/[p]\xi$. The factorization above defines the projection map:

$$q_*\colon E^*(BC_p) \cong E^*[\![\xi]\!]/[p]\xi \to E^*[\![\xi]\!]/\langle p\rangle\,\xi. \tag{2.2}$$

Quillen provides a formula (Equation 5.12) for $\chi^{m+n}P_{C_p,MU}(x)$, $m \gg 0$, when

$$x = [\mathbb{C}P^n] \in MU^{-2n}$$

and χ is defined by

$$\chi = \prod_{i=1}^{p-1}[i]\xi \in MU^{2(p-1)}[\![\xi]\!]/[p]\xi.$$

Proposition 2.3. *Suppose $f\colon MU \to E$ is a map of H_∞^2 ring spectra. Then there are power operations $P_{C_p,E}$ making the diagram below commute, and in particular*

$$f_*q_*\chi^{2n}P_{C_p,MU}[\mathbb{C}P^n] = q_*\chi^{2n}P_{C_p,E}f_*[\mathbb{C}P^n].$$

$$\begin{array}{ccccc}
MU^{2*} & \xrightarrow{P_{C_p,MU}} & MU^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_*\chi^{2n}} & MU^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
E^{2*} & \dashrightarrow^{P_{C_p,E}} & E^{2p*}[\![\xi]\!]/[p]\xi & \xrightarrow{q_*\chi^{2n}} & E^{2p*+4n(p-1)}[\![\xi]\!]/\langle p\rangle\xi
\end{array}$$

FIGURE 2.2. Complex orientations and power operations.

When E is BP and $f = r\colon MU \to BP$ is Quillen's map, we note that $r_*[\mathbb{C}P^n] = 0$ for $n \neq p^i - 1$. By considering the section $s\colon BP \to MU$, James McClure showed that Proposition 2.3 gives a necessary and sufficient condition for r to carry an H_∞^2 structure:

Theorem 2.4 ([BMMS86, VIII.7.7,7.8])**.** *The map $P_{C_p,BP} = r_*P_{C_p,MU}s_*$ is the only map that could possibly make Figure 2.2 commute*[1]*. Quillen's orientation is H_∞^2 if and only if the outer rectangle in this diagram commutes, and this occurs if and only if the elements*

$$MC_n(\xi) = r_*q_*\chi^{2n}P_{C_p,MU}[\mathbb{C}P^n] \in BP^{2n(p-2)}[\![\xi]\!]/\langle p\rangle\,\xi$$

are 0 when $n \neq p^i - 1$ for some i.

[1] The interested reader is encouraged to verify that commutativity of Figure 2.2 does not follow formally from the definition of $P_{C_p,BP}$.

In Theorem 4.16 we provide an alternate formulation of this result in the language of formal group laws. Later, in Section 5.4 we show that MC_n is trivial when $(p-1)$ does not divide n.

In Section 5 we obtain an explicit formula for MC_n, reducing our problem to algebra. Using this formula we obtain the calculations listed in Section 6 yielding Theorem 2.5, the computational backbone of the results in Section 1. The calculations are stated in terms of the Hazewinkel generators for $BP^* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, with $v_i \in BP^{-2(p^i-1)}$. We note that

$$r_*[\mathbb{C}P^{p-1}] = v_1.$$

Theorem 2.5. *For p satisfying the Small Prime Condition, we have the following expressions for* $MC_n \in BP^*[\![\xi]\!]/\langle p\rangle\xi$.

(1) *When* $p = 2$, $MC_2(\xi) = \left(v_1^6 + v_2^2\right)\xi^6 +$ *higher order terms*
and $MC_4(\xi) = v_1^4 v_2^2 \xi^{10} +$ *higher order terms.*
(2) *When* $p = 3$, $MC_4(\xi) = 2v_1^9\xi^{22} +$ *higher order terms.*
(3) *When* $p = 5$, $MC_8(\xi) = 3v_1^{16}\xi^{88} +$ *higher order terms.*
(4) *When* $p = 7$, $MC_{12}(\xi) = 4v_1^{22}\xi^{192} +$ *higher order terms.*
(5) *When* $p = 11$, $MC_{20}(\xi) = 9v_1^{34}\xi^{520} +$ *higher order terms.*
(6) *When* $p = 13$, $MC_{24}(\xi) = 11v_1^{40}\xi^{744} +$ *higher order terms.*

Since BP carries the universal p-typical orientation, the results of Theorem 2.5 can be applied to prove non-existence results for other p-typically oriented cohomology theories.

Proof of Theorem 1.3. Let $f\colon MU \to E$ be a map of H_∞ ring spectra as in Theorem 1.3, and let p be a prime satisfying the Small Prime Condition.

For $n \neq p^i - 1$, we must have $f_*[\mathbb{C}P^n] = 0$, since f factors through r. By assumption, Figure 2.2 commutes so $f_*(MC_n(\xi))$ must be 0 in the quotient ring. Equivalently, in $E^*[\![\xi]\!]$ we have

$$f_*(MC_n(\xi)) = g(\xi) \cdot \langle p\rangle\xi$$

for some $g(\xi)$. Since $\langle p\rangle\xi = p + \xi(\cdots)$ we see that the leading coefficient of $f_*(MC_n(\xi))$ must be divisible by p. Combining this fact with the calculations in Theorem 2.5, we will finish the proof by showing that E^*/p must be 0 and hence E^* must be a $\mathbb{Q}$-algebra.

For $p = 2$, the calculation for MC_2 shows $f(v_2)^2 = f(v_1)^6$ in $E^*/2$. Combining this with the calculation for MC_4 shows $f(v_1)^{10} = 0$ in $E^*/2$. However, by Landweber exactness, multiplication by $f(v_1)$ is an injection on $E^*/2$, so $E^*/2 = 0$.

For $p > 2$ the computation for $MC_{2(p-1)}$ implies that $f(v_1)$ is nilpotent in E^*/p. The same argument then shows that $E^*/p = 0$. □

3. E_∞ and H_∞ Ring Spectra

Let $\mathcal{S}$ denote the Lewis-May-Steinberger category of coordinate-free spectra and $h\mathcal{S}$ the stable homotopy category.

A spectrum in this category is indexed by finite dimensional subspaces of some countably infinite dimensional real inner product space $\mathcal{U}$. Let π be a subgroup of Σ_n, the symmetric group on n letters. The space of linear isometries $\mathcal{L}(\mathcal{U}^n, \mathcal{U})$ is a free contractible Σ_n-space and by restriction a free contractible π-space which we will denote $E\pi$.

For each subgroup π of Σ_n there is an extended power functor on unbased spaces, based spaces, and spectra. For an unbased space Z, a based space W, and a spectrum X, the definitions are

$$D_\pi Z = E\pi \times_\pi Z^{\times n}$$
$$D_\pi W = E\pi_+ \wedge_\pi W^{\wedge n}$$
$$D_\pi X = E\pi \ltimes_\pi X^{\wedge n}.$$

where $\ltimes$ is the twisted half-smash product of [LMS86]. The functor from unbased to based spaces given by adjoining a disjoint basepoint relates the extended cartesian power on unbased spaces and the extended smash power on based spaces. For an unbased space Z, there is a homeomorphism of based spaces,

$$D_\pi(Z_+) \cong (D_\pi Z)_+.$$

We will be using power operations on unreduced cohomology theories; as a consequence we will focus on unbased rather than based spaces. The extended Cartesian power on unbased spaces is related to the extended smash power on spectra by the following: For an unbased space Z

$$D_\pi \Sigma^\infty_+(Z) = D_\pi \Sigma^\infty(Z_+) \cong \Sigma^\infty D_\pi(Z_+) \cong \Sigma^\infty(D_\pi Z)_+ = \Sigma^\infty_+ D_\pi Z. \tag{3.1}$$

With Equation 3.1 in mind, we may implicitly apply the functor Σ^∞_+ and will use the notation $D_\pi Z$ to denote either an unbased space or a spectrum, as determined by context.

Definition 3.2. Let D be the functor on $\mathcal{S}$ such that

$$DX = \bigvee_{n\geq 0} D_{\Sigma_n} X.$$

The following result is standard (for example, see [Rez98]).

Proposition 3.3. *There are natural transformations*

$$\mu\colon D^2 \to D$$
$$\eta\colon Id \to D$$

that make D a monad on $\mathcal{S}$.

Definition 3.4. The category of E_∞ ring spectra is the category of D-algebras in $\mathcal{S}$.

Proposition 3.5. *The monad D on $\mathcal{S}$ descends to a monad $\widetilde{D}$ on the stable homotopy category $h\mathcal{S}$.*

Proof. In [LMS86] it is shown that this functor preserves homotopy equivalences between cell spectra and takes cellular spectra to cellular spectra. It follows that D has a well-defined functor on the stable homotopy category, modeled by cellular spectra with homotopy classes of maps and that the structure maps above pass to the stable category. □

Definition 3.6. The category of H_∞ ring spectra is the category of $\widetilde{D}$-algebras in $h\mathcal{S}$.

Proposition 3.7. *Let $\Gamma\colon \mathcal{S} \to h\mathcal{S}$ denote the canonical functor. If X is an E_∞ ring spectrum, then ΓX is an H_∞ ring spectrum.*

Remark 3.8. Nearly all known H_∞ ring spectra arise by applying Γ to an E_∞ ring spectrum. In [Noe09] the second author provides an example of one that is not.

Definition 3.9. Suppose X is a spectrum, E is an H_∞ ring spectrum, and $f\colon X \to E$ is a map representing a cohomology class in $E^0(X)$. Define the π^{th} external cohomology operation

$$\mathcal{P}_{\pi,E}\colon E^0(X) \to E^0(D_\pi X)$$

by

$$(X \xrightarrow{f} E) \mapsto (D_\pi X \xrightarrow{D_\pi f} D_\pi E \to D_{\Sigma_n} E \hookrightarrow DE \xrightarrow{\mu} E).$$

If Y is a space, $Y^{\times n}$ is equipped with the π action induced by the inclusion $\pi \to \Sigma_n$. Regarding Y as a trivial π-space, the diagonal map

$$\Delta\colon Y \to Y^{\times n}$$

is π-equivariant.

Definition 3.10. Suppose Y is a space and E is an H_∞ ring spectrum. Define $\delta\colon B\pi \times Y \to D_\pi Y$ as the following composite:

$$\delta\colon (B\pi \times Y) \simeq E\pi \times_\pi Y \xrightarrow{E\pi\times\Delta} E\pi \times_\pi Y^n \cong D_\pi Y.$$

Define the π^{th} internal cohomology operation $P_{\pi,E}\colon E^0(Y) \to E^0(B\pi \times Y)$ as the composite

$$E^0(Y) \xrightarrow{\mathcal{P}_{\pi,E}} E^0(D_\pi Y) \xrightarrow{\delta^*} E^0(B\pi \times Y).$$

Notation 3.11. We will drop the subscript E from the power operations $\mathcal{P}_{\pi,E}$ and $P_{\pi,E}$, when it is clear from the context.

3.1. **H_∞^d ring spectra.** An H_∞^d ring structure on a spectrum E is a compatible family of maps

$$D_{\Sigma_n}\Sigma^{di}E \to \Sigma^{din}E$$

for all $i \in \mathbb{Z}$ [BMMS86, I.4.3]. When $i = 0$, these maps define an H_∞ structure on E, so every H_∞^d ring spectrum is an H_∞ ring spectrum. The compatibility conditions are graded analogs of those for an H_∞ ring spectrum, and an H_∞^d structure on E determines an H_∞ structure on the infinite wedge[2]

$$\bigvee_{i\in\mathbb{Z}} \Sigma^{di}E.$$

Maps of H_∞^d ring spectra are those which commute with the family of structure maps and so the category of H_∞^d ring spectra is a subcategory of the category of H_∞ ring spectra.

Suppose Y is a space and E is an H_∞^d ring spectrum. For each $\pi \leqslant \Sigma_n$ and for each integer i, we have the following power operations:

$$\mathcal{P}_{\pi,E}\colon E^{di}(Y) \to E^{din}(D_\pi Y)$$
$$P_{\pi,E}\colon E^{di}(Y) \to E^{din}(B\pi \times Y).$$

When $i = 0$, these maps are simply the above power operations defined using the underlying H_∞ structure on E.

[2] We note that the argument for the converse to this statement, given in [BMMS86, II.1.3], is incorrect. We were unable to find a proof for the converse to hold in this generality.

3.2. **The Thom isomorphism and H_∞^2 orientations.** Let V_k denote the standard representation of Σ_k on $\mathbb{C}^k$ and $B\Sigma_k^{V_k \otimes \mathbb{C}^i}$ be the Thom spectrum of the complex vector bundle $V_k \otimes \mathbb{C}^i$ over $B\Sigma_k$. Recall [LMS86, Ch. X] that

$$D_{\Sigma_k} S^{2i} \cong B\Sigma_k^{V_k \otimes \mathbb{C}^i}. \tag{3.12}$$

Since $V_k \otimes \mathbb{C}^i$ is a complex vector bundle, for any complex oriented cohomology theory E we have a Thom isomorphism

$$E^*(\Sigma^{2ki} B\Sigma_k) \cong E^*(B\Sigma_k^{V_k \otimes \mathbb{C}^i}).$$

Taking $\mu_{i,k}$ to be a map representing the Thom class, the Thom isomorphism yields the following commutative diagram. The horizontal map is induced by the natural inclusion $S^{2ki} \to D_{\Sigma_k} S^{2i}$ and e is the unit $S \to E$.

$$\begin{array}{ccc}
S^{2ki} & \longrightarrow & D_{\Sigma_k} S^{2i} \\
& {\scriptstyle \Sigma^{2ki} e} \searrow \quad \swarrow {\scriptstyle \mu_{i,k}} & \\
& \Sigma^{2ki} E. &
\end{array}$$

Note that although the Thom classes $\mu_{i,k}$ clearly depend on the cohomology theory E, we will abuse notation and use the same symbol regardless of the cohomology theory.

When $E = MU$, McClure shows [BMMS86, VII] that the $\mu_{i,k}$ combine with the H_∞ structure maps

$$\mu_k \colon D_{\Sigma_k} MU \to DMU \xrightarrow{\mu} MU$$

to define an H_∞^2 structure for MU: The structure maps are those given by the top horizontal composite in Figure 3.1.

$$\begin{array}{ccccccc}
D_{\Sigma_k}(\Sigma^{2i} MU) & \longrightarrow & D_{\Sigma_k} S^{2i} \wedge D_{\Sigma_k} MU & \xrightarrow{\mu_{i,k} \wedge \mu_k} & \Sigma^{2ki} MU \wedge MU & \longrightarrow & \Sigma^{2ki} MU \\
\downarrow {\scriptstyle D_{\Sigma_k}(f)} & & \downarrow {\scriptstyle D_{\Sigma_k} S^{2i} \wedge f} & & \downarrow {\scriptstyle \Sigma^{2ki} f \wedge f} & & \downarrow {\scriptstyle \Sigma^{2ki} f} \\
D_{\Sigma_k}(\Sigma^{2i} E) & \longrightarrow & D_{\Sigma_k} S^{2i} \wedge D_{\Sigma_k} E & \xrightarrow{\mu_{i,k} \wedge \mu_k} & \Sigma^{2ki} E \wedge E & \longrightarrow & \Sigma^{2ki} E.
\end{array}$$

FIGURE 3.1. H_∞^2 orientations.

In this way the Thom isomorphism for complex oriented theories gives an equivalence between H_∞ orientations and H_∞^2 orientations.

Theorem 3.13. *An orientation $MU \to E$ is H_∞ if and only if it is H_∞^2.*

Proof. By neglect of structure every H_∞^2 orientation is H_∞. Consider an H_∞ complex orientation $f \colon MU \to E$. Figure 3.1 is induced by this structure and the left and right squares in this diagram commute for any orientation on E. The center square is the smash product of the following two squares:

$$\begin{array}{ccc}
D_{\Sigma_k} S^{2ik} & \xrightarrow{\mu_{i,k}} & \Sigma^{2ki} MU \\
\| & & \downarrow {\scriptstyle \Sigma^{2ki} f} \\
D_{\Sigma_k} S^{2ik} & \xrightarrow{\mu_{i,k}} & \Sigma^{2ki} E
\end{array}
\qquad\qquad
\begin{array}{ccc}
D_{\Sigma_k} MU & \xrightarrow{\mu_k} & MU \\
\downarrow {\scriptstyle f} & & \downarrow {\scriptstyle \Sigma^{2ki} f} \\
D_{\Sigma_k} E & \xrightarrow{\mu_k} & E
\end{array}$$

The left square commutes since f sends MU-Thom classes to E-theory Thom classes. The right square commutes since f is an H_∞ ring map.

It follows that the center square and therefore the entire diagram commutes in Figure 3.1. Another elementary diagram chase, using the H^2_∞ structure of MU, shows that the bottom horizontal composite defines an H^2_∞ structure on E. □

4. The Formal Group Law Condition

4.1. **Formal group laws.** We recall some well-known facts about complex-oriented cohomology theories and formal group laws (for example, see [Ada95, Part II] or [Rav00]).

Definition 4.1. A (commutative, 1-dimensional) formal group law F over a commutative ring k is a connected bicommutative, associative, topological Hopf algebra $\mathcal{A}$ with a specified isomorphism $\mathcal{A} \cong k[\![x]\!]$.

By forgetting the grading, a graded Hopf algebra of the above form is a formal group law. For such Hopf algebras the completed tensor product provides the following isomorphism:

$$\mathcal{A}\widehat{\otimes}\mathcal{A} \cong k[\![x_1, x_2]\!].$$

Definition 4.2. Given a ring map $f\colon k \to k'$ and a formal group law $\mathcal{A}$ over k, the push-forward of $\mathcal{A}$ along f is the formal group law $\mathcal{A}\widehat{\otimes}^f_k k'$ over k'.

One can formally define a ring L and a formal group law $\mathcal{A}$ over L such that

$$\mathcal{R}ing(L, k) \cong \text{Formal group laws over } k \tag{4.3}$$

$$f \to \mathcal{A}\widehat{\otimes}^f_L k \tag{4.4}$$

Notation 4.5. We will identify a formal group law F with the formal power series:

$$x_1 +_F x_2 = \Delta(x) \in k[\![x_1, x_2]\!].$$

Definition 4.6. Given a commutative ring k, we formally adjoin the q^{th} roots of unity. A formal group law F over k is p-typical, if for all primes $q \neq p$, the formal sum over the q^{th} roots of unity

$$\sum_{\zeta^q=1}^F \zeta x$$

is trivial.

4.2. **Connection to complex orientations.** Recall that if X is a space and E is a spectrum, the function spectrum

$$E^X = F(\Sigma^\infty_+ X, E)$$

defines a cohomology theory satisfying

$$E^{X,*}(Y) \cong E^*(X \times Y), \tag{4.7}$$

for every space Y. Moreover, if E admits the structure of a ring spectrum (or an H_∞ ring spectrum) then so does E^X.

Proposition 4.8 ([Lan76, 3.1])**.** *The spectra MU^{BC_p} and BP^{BC_p} are ring spectra satisfying the following natural isomorphisms:*

$$MU^{BC_p,*}X \cong MU^*(BC_p)\widehat{\otimes}_{MU_*}MU^*(X)$$
$$BP^{BC_p,*}X \cong BP^*(BC_p)\widehat{\otimes}_{BP_*}BP^*(X).$$

In complex cobordism there is a tautological element x giving an isomorphism

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!],$$

and we fix an element ξ such that

$$MU^*(BC_p) \cong MU^*[\![\xi]\!]/[p]\xi.$$

Hence we have

$$MU^{BC_p,*}(\mathbb{C}P^\infty) \cong MU^*[\![\xi,\, x]\!]/[p]\xi.$$

An orientation $f\colon MU \to E$ fixes generators x and ξ in E-cohomology that define analogous isomorphisms.

The above tautological isomorphism in complex cobordism combined with the multiplication on $\mathbb{C}P^\infty$ classifying a tensor product of line bundles defines a formal group law over MU^*. An orientation $MU \to E$, induces a map $MU^* \to E^*$ which defines a formal group law structure (also denoted by E) on $E^*(\mathbb{C}P^\infty)$ by pushing forward the formal group law on MU, or equivalently [Ada95, II.4.6], by fixing the generator $x \in E^*(\mathbb{C}P^\infty)$ above.

Theorem 4.9 ([Qui69])**.** *The map*

$$L \to MU^*$$

classifying the tautological formal group law over MU^* is an isomorphism.

Rationally, we can describe this isomorphism explicitly in terms of the cobordism classes

$$[\mathbb{C}P^n] \in MU^{-2n}.$$

Proposition 4.10. *There is an algebra isomorphism*

$$MU^* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \dots].$$

With these choices, the power operation

$$P_{C_p,MU}\colon MU^{2*}(\mathbb{C}P^\infty) \to MU^{BC_p,\,2p*}(\mathbb{C}P^\infty)$$

of Figure 2.1 on the generator x is given by the following formula [Qui71, Prop. 3.17]:

$$P_{C_p,MU}(x) = \prod_{i=0}^{p-1} \big([i]\xi +_{MU} x\big). \tag{4.11}$$

Of course, after applying an orientation $f\colon MU \to E$ we obtain

$$f_* P_{C_p,MU}(x) = \prod_{i=0}^{p-1} \big([i]\xi +_{E} x\big). \tag{4.12}$$

Considering Equation 4.12 as a power series in x whose coefficients are power series in ξ, we define

$$a_i \equiv a_i(\xi) \in E^{2(p-i-1)}(BC_p) \cong E^{2(p-i-1)}[\![\xi]\!]/[p]\xi, \quad \text{for} \quad i \geq 0$$

by the following expansion:

$$f_* P_{C_p,MU}(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots . \tag{4.13}$$

By pulling back along the inclusion

$$S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty,$$

and applying the C_p analogue of Equation 3.12 we see that $a_0 x$ is the Euler class of the regular representation of C_p and

$$a_0 = \chi, \tag{4.14}$$

is the Euler class of the *reduced* regular representation of C_p.

The next result follows immediately from Proposition 5.11.

Proposition 4.15. *Let X be a topological space and let*

$$\overline{P_{C_p}} \colon MU^{2*}(X) \to MU^{BC_p,2*}(X)[\chi^{-1}]$$

be the map which in degree $2n$ is P_{C_p}/χ^n. Then $\overline{P_{C_p}}$ and $r_*\overline{P_{C_p}}$ are maps of graded rings.

Using this result and the discussion preceding Theorem 4.9 we see that the maps $\overline{P_{C_p,MU}}$ and $r_* \circ \overline{P_{C_p,MU}}$ define formal group laws $\mathcal{UP}$ and $\mathcal{VP}$ over $MU^{BC_p}[\chi^{-1}]$ and $BP^{BC_p}[\chi^{-1}]$ respectively.

$$\begin{array}{ccc} MU^{2*}(\mathbb{C}P^\infty) & \xrightarrow{\overline{P_{C_p,MU}}} & MU^{BC_p,2*}[\chi^{-1}](\mathbb{C}P^\infty) \\ {\scriptstyle r_*}\downarrow & & {\scriptstyle r_*}\downarrow \\ BP^{2*}(\mathbb{C}P^\infty) & \overset{\overline{P_{C_p,BP}}}{\dashrightarrow} & BP^{BC_p,2*}[\chi^{-1}](\mathbb{C}P^\infty) \end{array}$$

Figure 4.1. A formal group theoretic condition.

Theorem 4.16. *The map $r\colon MU \to BP$ is a map of H_∞ ring spectra if and only if $\mathcal{VP}$ is p-typical.*

Proof. Since the map r is a p-universal orientation of BP, there exists a map

$$P\colon BP \to BP^{BC_p}[\chi^{-1}].$$

that makes Figure 4.1 commute if and only if $\mathcal{VP}$ is p-typical. This happens if and only if the indecomposables in MU^{-2n} map to zero under $\overline{P_{C_p,MU}}$ when $n \neq p^i - 1$. Since the cobordism classes $[\mathbb{C}P^n]$ are rationally polynomial generators and all rings in sight are torsion-free, we see that $\mathcal{VP}$ is p-typical if and only if the elements MC_n of Theorem 2.4 map to 0. □

5. Computing the Obstructions

Before proving Proposition 5.21 we will need some notation.

5.1. **Notation.** Throughout this paper, the symbol

$$\alpha = (\alpha_0, \alpha_1, \dots) \tag{5.1}$$

with $\alpha_n = 0$ for $n \gg 0$, will be a multi-index beginning with α_0.

As the reader will see, it will also be convenient to have notation for multi-indices starting with α_1, so we set

$$\overline{\alpha} = (\alpha_1, \alpha_2, \dots). \tag{5.2}$$

Given an infinite list of variables $a_0, a_1, a_2, \dots$, we set

$$a^\alpha = a_0^{\alpha_0} a_1^{\alpha_1} \dots \quad \text{and} \quad a^{\overline{\alpha}} = a_1^{\alpha_1} a_2^{\alpha_2} \dots. \tag{5.3}$$

For any integer n we define the modified multinomial coefficient $\mu(n;\overline{\alpha})$ by the formal power series expansion:

$$(1+b_1+b_2\cdots)^n=\sum_{\overline{\alpha}}\mu(n;\overline{\alpha})b^{\overline{\alpha}}. \tag{5.4}$$

We also set:

$$|\alpha|=\sum_{i\geq 0}\alpha_i \tag{5.5}$$

$$|\alpha|'=\sum_{i\geq 0}i\alpha_i=|\overline{\alpha}|'. \tag{5.6}$$

Given a formal power series $S(z)$, let

$$S(z)[z^k]=\text{coefficient of } z^k \text{ in } S(z). \tag{5.7}$$

5.2. **Additive and multiplicative operations.** Recall that the Landweber-Novikov algebra is the subalgebra of MU^*MU whose elements define additive cohomology operations. This algebra is a free $\mathbb{Z}_{(p)}$-module on elements

$$s_{\alpha_1,\alpha_2,\ldots}=s_{\overline{\alpha}} \tag{5.8}$$

dual to the standard basis

$$t_1^{\alpha_1}t_2^{\alpha_2}\cdots=t^{\overline{\alpha}}\in MU_{2|\overline{\alpha}|'}MU\cong MU_{2|\overline{\alpha}|'}BU. \tag{5.9}$$

To simplify our formulas we extend the indexing to multi-indices starting with α_0 by setting

$$s_\alpha\equiv s_{\overline{\alpha}}\in MU^{2|\alpha|'}MU. \tag{5.10}$$

Proposition 5.11 ([Qui71]). *If $x\in MU^{-2q}(X)$ and $m\gg 0$ then*

$$\chi^{m+q}P_{C_p}x=\sum_{|\alpha|=m}a^\alpha s_\alpha(x). \tag{5.12}$$

Since the right hand side of Equation 5.12 is additive in x and P_{C_p} is always multiplicative, we obtain Proposition 4.15 by inverting χ.

For any complex oriented cohomology theory E,

$$[i]\xi+_E x\equiv i\xi \mod x,$$

which implies

$$\chi=a_0\equiv(p-1)!\xi^{p-1} \mod \xi^p. \tag{5.13}$$

It follows that inverting χ factors through inverting ξ, so when E is MU or BP, we have:

$$E^{BC_p,*}(X)[\chi^{-1}]\cong E^*(X)[\![\xi]\!][\chi^{-1}]/[p]\xi\cong E^*(X)[\![\xi]\!][\chi^{-1}]/\langle p\rangle\xi.$$

Since

$$\langle p\rangle\xi=[p]\xi/\xi\equiv p \mod \xi$$

and $(p-1)!$ is not divisible by p, $q_*\chi$ is not a zero-divisor. It follows, when $E=MU$ or BP, that the localization map

$$E^*(X)[\![\xi]\!]/\langle p\rangle\xi\to E^*(X)[\![\xi]\!][\chi^{-1}]/\langle p\rangle\xi$$

is an injection. Applying Proposition 4.15 proves the following: Proposition 4.15.

Proposition 5.14. *The composites*

$$q_* P_{C_p} \colon MU^*(\mathbb{C}P^\infty) \to MU^{BC_p,*}(\mathbb{C}P^\infty)/\langle p\rangle\xi$$
$$r_* q_* P_{C_p} \colon MU^*(\mathbb{C}P^\infty) \to BP^{BC_p,*}(\mathbb{C}P^\infty)/\langle p\rangle\xi$$

are ring maps.

5.3. **Derivation of MC_n.** We begin with the following refinement of Equation 5.12:

Lemma 5.15.

$$\chi^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} a^\alpha s_\alpha[\mathbb{C}P^n]. \tag{5.16}$$

Proof. By Equation 5.12, for $k \gg 0$ we have:

$$\begin{aligned}
\chi^{2n+k} P_{C_p}[\mathbb{C}P^n] &= \sum_{|\alpha|=n+k} a^\alpha s_\alpha[\mathbb{C}P^n] \\
&= \sum_{\alpha_0=0}^{n+k} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a_0^{\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= \sum_{\alpha_0=0}^{k-1} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] + \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n]
\end{aligned}$$

Since MU^* is concentrated in non-positive degrees,

$$s_{\overline{\alpha}}([\mathbb{C}P^n]) \in MU^{2|\overline{\alpha}|'-2n} = 0$$

when $|\overline{\alpha}|' > n$.

In the first sum of the last equation, $|\overline{\alpha}| > n$. Since

$$|\overline{\alpha}|' = \sum_{i\geq 1} i\alpha_i \geq \sum_{i\geq 1} \alpha_i = |\overline{\alpha}|,$$

all terms in the first sum are trivial. This leaves us with

$$\begin{aligned}
\chi^{2n+k} P_{C_p}[\mathbb{C}P^n] &= \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= a_0^k \sum_{\alpha_0=0}^{n} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= a_0^k \sum_{|\alpha|=n} a^\alpha s_\alpha[\mathbb{C}P^n].
\end{aligned}$$

Since $a_0 = \chi$ is a not a zero-divisor the lemma follows. ∎

Theorem 5.17 ([Ada95, I.8.1])**.**

$$s_\alpha[\mathbb{C}P^n] = \mu(-(n+1); \overline{\alpha})[\mathbb{C}P^{n-|\alpha|'}] \tag{5.18}$$

We combine Equations 5.16 and 5.18 and obtain:

Theorem 5.19.

$$MC_n(\xi) \equiv r_* q_* \chi^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} \mu(-(n+1);\overline{\alpha})\ r_*[\mathbb{C}P^{n-|\alpha|'}]\ a^\alpha.$$

Remark 5.20. After correcting a couple of typographical errors, this is a simplified version of the formula given in [BMMS86, VIII.7.8].

For $n \neq p^i - 1$, the power series $MC_n(\xi)$ are McClure's obstructions to the existence of H_∞ strucure on Quillen's map $r\colon MU \to BP$. Note that, if $i+1$ is not a power of p then $r_*[\mathbb{C}P^i] = 0$, so many of the summands on MC_n are zero. For our calculations, we make use of the following alternate expression:

Proposition 5.21. *McClure's formula is equivalent to*

$$MC_n(\xi) = \chi^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k].$$

Proof. We rearrange the sum by summing over $|\alpha|' = k$. Now the condition $|\alpha| = n$ is simply a constraint on α_0.

$$\begin{aligned} MC_n(\xi) &= \sum_{k=0}^{n} \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}} \mu(-(n+1);\overline{\alpha})\ r_*[\mathbb{C}P^{n-|\alpha|'}]\ a^\alpha \\ &= \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}} \mu(-(n+1);\overline{\alpha})\ a^\alpha. \end{aligned}$$

To simplify the inner sum, we consider the following formal series and use the definition of the modified multinomial coefficients given in Equation 5.4:

$$\begin{aligned} a_0^{2n+1} \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} &= a_0^n \left(1 + \frac{a_1}{a_0} z + \frac{a_2}{a_0} z^2 + \cdots\right)^{-(n+1)} \\ &= a_0^n \sum_{\overline{\alpha}} \mu(-(n+1);\overline{\alpha}) \left(\frac{a_1}{a_0} z\right)^{\alpha_1} \left(\frac{a_2}{a_0} z^2\right)^{\alpha_2} \cdots \\ &= \sum_{\overline{\alpha}} \mu(-(n+1);\overline{\alpha})\ \frac{a_0^n\ a_1^{\alpha_1} a_2^{\alpha_2} \cdots}{a_0^{\alpha_1+\alpha_2+\cdots}}\ z^{\alpha_1+2\alpha_2+\cdots} \\ &= \sum_{k\geq 0} z^k \left(\sum_{|\overline{\alpha}|'=k} \mu(-(n+1);\overline{\alpha})\ \frac{a_0^n\ a_1^{\alpha_1}\ a_2^{\alpha_2} \cdots}{a_0^{\alpha_1+\alpha_2+\cdots}}\right) \end{aligned}$$

Now we consider the coefficients of z^k. For $k \leq n$, the restriction $|\overline{\alpha}|' = k$ implies $|\overline{\alpha}| \leq n$. Hence we may extend to a sum over multi-indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ with $\alpha_0 = n - |\overline{\alpha}|$ which forces $|\alpha| = n$. Thus we have, for $0 \leq k \leq n$,

$$a_0^{2n+1} \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k] = \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}} \mu(-(n+1);\overline{\alpha})\, a^\alpha.$$

□

5.4. **Sparseness.** In this section we prove that, at odd primes, many of the MC_n do in fact vanish. We also give a sparseness result for the a_i.

Proposition 5.22. *If $n \not\equiv 0 \mod p-1$ then $MC_n = 0$.*

Proof. The statement is vacuously true at the prime 2, so assume p is odd. The summands of the equation in Theorem 5.19 are constant multiples of $r_*[\mathbb{C}P^i]$ and a^α. The first term is nonzero only in degrees divisible by $2(p-1)$ and it follows from the lemma below that the nonzero a^α are also concentrated in degrees divisible by $2(p-1)$.

Now the left side of the equation in Theorem 5.19 is in degree $2n(p-2)$ and the right hand side is concentrated in degrees divisible by $2(p-1)$. Since 2 and $(p-2)$ are units mod p we see that MC_n can only be non-zero when n is divisible by $p-1$. □

Lemma 5.23. *The elements $a_i \in BP^*(BC_p)$ defined in Equation 4.13 are zero if $i \not\equiv 0 \mod p-1$.*

Proof. Since the lemma is vacuously true for $p = 2$, we will assume p is odd.

The action of $C_p^\times$ on C_p induces an action of $C_p^\times$ on BC_p. In $BP^*(BC_p)$, an element $v \in C_p^\times$ acts on $[i]\xi$ by

$$[i]\xi \mapsto [vi]\xi.$$

Since the product

$$\prod_{i=1}^{p-1}([i]\xi +_{BP} x)$$

is invariant under this action, we see that $a_i \in BP^{2(p-i-1)}(BC_p)^{C_p^\times}$.

The Atiyah-Hirzebruch spectral sequence computing $BP^*(BC_p)$ collapses at the E_2 page, which is of the form $H^*(BC_p, BP^*)$. The group action above induces a group action on this page. Since the edge homomorphism $BP^*(BC_p) \to H^*(BC_p)$, is an equivariant surjection that restricts to an isomorphism along the 0th row, the associated graded of $BP^*(BC_p)^{C_p^\times}$ is isomorphic to $H^*(BC_p)^{C_p^\times} \otimes BP^* \cong \mathbb{Z}/p[\xi^{p-1}] \otimes BP^*$.

Since this last group is concentrated in degrees divisible by $2(p-1)$, if $a_i \neq 0$ then

$$a_i \in BP^{2(p-1)*}(BC_p).$$

The congruence

$$\frac{|a_i|}{2} = (p-1-i) \equiv i \equiv 0 \mod (p-1)$$

implies i is divisible by $p-1$.

□

As a result, it is of interest to consider $MC_{2(p-1)}$. In this case, one can give the formula more explicitly:

$$\begin{aligned} MC_{2(p-1)}(\xi) &= a_0^{2p-4} r_*[\mathbb{C}P^{(p-1)}]\left(-(2p-1)a_0a_{(p-1)}\right) \\ &+ a_0^{2p-4} r_*[\mathbb{C}P^0]\left(-(2p-1)a_0a_{2(p-1)} + p(2p-1)a_{(p-1)}^2\right) \end{aligned} \tag{5.24}$$

Making the simplifications $[\mathbb{C}P^0] = 1$ and $r_*[\mathbb{C}P^{p-1}] = v_1$, we have

$$MC_{2(p-1)}(\xi) = (2p-1)a_0^{2p-4}\left(-v_1a_0a_{(p-1)} - a_0a_{2(p-1)} + pa_{(p-1)}^2\right)$$

6. Calculations

In this section, we outline the computation of the MC_n, work through an example at the prime 2, and display results at the primes $p \leq 13$. We have developed a Sage package [JN10] to automate the calculations.

6.1. **Description of calculation.** We are working in $BP^*[\![\xi]\!]/\langle p\rangle\xi$, and we emphasize reduction modulo $\langle p\rangle\xi$ by writing $\equiv \bmod \langle p\rangle\xi$ instead of equality. Our calculations have three parameters: the prime, p, the value of n, and a truncation number, k. All of our computations are modulo $(\xi, x)^{k+1}$. If power series $f(\xi)$ and $g(\xi)$ are equal modulo the ideal $(\xi)^{k+1}$, we write

$$f(\xi) = g(\xi) + O(\xi)^{k+1}.$$

It is important to note, because of this choice, that the range of accurate coefficients for the $a_i(\xi)$ decreases as i grows. Each a_i is accurate modulo ξ^{k-i+1}. Using the formula above, and the fact that $a_0 = (p-1)! \cdot \xi^{p-1} + \cdots$, we see that $MC_{2(p-1)}$ is accurate modulo ξ^{k-p+2}.

We have made efforts to streamline the computation, but our results are limited by the computational complexity of formal group law calculations. Determining the series $\exp_{BP}$ is already a task whose computation time grows quickly with the length of the input. Calculating the a_i is also a high-complexity task, and as a result we do not expect direct computation to be a feasible approach for large primes. We have not been able to work in a large enough range to detect non-zero values of MC_n for primes greater than 13.

To check for triviality modulo $\langle p\rangle\xi$, we make use of the following reduction algorithm: Suppose $g \in (\xi)^m \subset BP^*[\![\xi]\!]$ and write

$$g = \sum_{i\geq 0} g_i \xi^{i+m}$$

with $g_i \in BP^*$ and $g_0 \neq 0$. If $p \not| g_0$ then $g \notin (\langle p\rangle\xi)$. If $g_0 = p \cdot d_0$ for $d_0 \in BP^*$, then we have

$$g'(\xi) = g(\xi) - d_0 \xi^m \langle p\rangle\xi$$

with $g' \in (\xi)^{m+1}$ and $g \equiv g' \bmod \langle p\rangle\xi$. Iterating this process converges in the ξ-adic topology.

A similar adaptation of the usual Euclidean algorithm for division by p gives the following. We state the result integrally since we are working with the Hazewinkel generators throughout.

Proposition 6.1 (Division Algorithm)**.** *Let g be a power series in $\mathbb{Z}[v_1, v_2, \ldots][\![\xi]\!]$ and let $\langle p\rangle\xi$ be the reduced p-series, computed using the Hazewinkel generators. Then there are unique power series d and $s = \sum_{i\geq 0} s_i\xi^i$ in $\mathbb{Z}[v_1, v_2, \ldots][\![\xi]\!]$ such that*

$$g(\xi) = d \cdot \langle p\rangle\xi + s$$

and such that the polynomials $s_i \in \mathbb{Z}[v_1, v_2, \ldots]$ have coefficients in the range $\{0, \ldots, p-1\}$. The series g is divisible by $\langle p\rangle\xi$ if and only if $s = 0$.

6.2. **Sample calculation, $p = 2$.** To give the reader a sense of how these calculations are implemented, we work through the calculation of $MC_2(\xi)$ with the minimum range of coefficients necessary to see that it is non-zero. For this, it is necessary to work modulo $(x, \xi)^8$. The formula for MC_2 is given in Proposition 5.21:

$$MC_2(\xi) = a_0^5 \sum_{k=0}^{2} r_*[\mathbb{C}P^{n-k}] \cdot \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k].$$

Now one can easily check the formal computation

$$\left(\sum_{i\geq 0} a_i z^i\right)^{-1} = a_0^{-1} - a_1 a_0^{-2} z + \left(-a_2 a_0^{-2} + a_1^2 a_0^{-3}\right) z^2 + O(z)^3$$

and hence

$$\left(\sum_{i\geq 0} a_i z^i\right)^{-3} = a_0^{-3} - 3a_1 a_0^{-4} z + \left(-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5}\right) z^2 + O(z)^3.$$

The image of $[\mathbb{C}P^i] \in MU^{-2i}$ under r_* is given by

$$r_*[\mathbb{C}P^i] = \begin{cases} 0 & \text{if } i \neq p^k - 1 \\ [\mathbb{C}P^i] = p^k \ell_k & \text{if } i = p^k - 1 \end{cases}$$

The elements ℓ_k are rational generators for BP, but it is convenient to work with integral generators. For this example we choose the Hazewinkel generators v_k, but the result is independent of this choice. It will be necessary only to use $v_1 = 2\ell_1$, so we work modulo the ideal $I = (v_2, v_3, \ldots)$. Modulo I we have $4\ell_2 = v_1^3$, and this will be the only additional substitution we need to use.

Returning to the calculation, we have

$$[\mathbb{C}P^0] = 1, \quad r_*[\mathbb{C}P^1] = 2\ell_1 = v_1, \quad \text{and } r_*[\mathbb{C}P^2] = 0$$

and so

$$\begin{aligned} MC_2(\xi) &= a_0^5 \left(-3v_1 a_0^{-4} a_1 + (-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5})\right) \\ &= 6a_1^2 - 3a_0 a_2 - 3v_1 a_0 a_1. \end{aligned}$$

To continue, we determine $a_0(\xi)$, $a_1(\xi)$, and $a_2(\xi)$. These are defined by the following (c.f. 4.12, 4.13):

$$\begin{aligned} P_{C_p,BP}(x) = r_* P_{C_p,MU}(x) = \prod_{i=0}^{1} ([i]\xi +_{BP} x) &= x \cdot \exp\left(\log(\xi) + \log(x)\right) \\ = x \cdot \big[& a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 \\ & + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \\ & + O(x,\xi)^8 \ \big]. \end{aligned}$$

The logarithm is

$$\log_{BP}(\xi) = \xi + \ell_1 \xi^2 + \ell_2 \xi^4 + O(\xi)^8$$

and hence the exponential is

$$\begin{aligned} \exp_{BP}(\xi) = \xi - \ell_1 \xi^2 + 2\ell_1^2 \xi^3 &+ \left(-5\ell_1^3 - \ell_2\right) \xi^4 \\ &+ \left(14\ell_1^4 + 6\ell_1 \ell_2\right) \xi^5 \\ &+ \left(-42\ell_1^5 - 28\ell_1^2 \ell_2\right) \xi^6 \\ &+ \left(132\ell_1^6 + 120\ell_1^3 \ell_2 + 4\ell_2^2\right) \xi^7 \\ &+ O(\xi)^8. \end{aligned}$$

Using the logarithm and exponential, we give the reduced 2-series:

$$\begin{aligned}\langle 2\rangle\xi = \frac{1}{\xi}\exp(2\log(\xi)) &= 2 - 2\ell_1\xi + 8\ell_1^2\xi^2 \\ &+ \left(-36\ell_1^3 - 14\ell_2\right)\xi^3 \\ &+ \left(176\ell_1^4 + 120\ell_1\ell_2\right)\xi^4 \\ &+ \left(-912\ell_1^5 - 888\ell_1^2\ell_2\right)\xi^5 \\ &+ \left(4928\ell_1^6 + 6240\ell_1^3\ell_2 + 448\ell_2^2\right)\xi^6 \\ &+ O(\xi)^7\end{aligned}$$

Substituting the Hazewinkel generators, and working modulo v_2,

$$\begin{aligned}\langle 2\rangle\xi &= 2 - v_1\xi + 2v_1^2\xi^2 \\ &- 8v_1^3\xi^3 \\ &+ 26v_1^4\xi^4 \\ &- 84v_1^5\xi^5 \\ &+ 300v_1^6\xi^6 \\ &+ O(\xi)^7\end{aligned}$$

and

$$\begin{aligned}P_{C_p,BP} = x\cdot\big[&\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right) \\ &- \ell_1\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^2 \\ &+ 2\ell_1^2\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^3 \\ &+ \left(-5\ell_1^3 - \ell_2\right)\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^4 \\ &+ \left(14\ell_1^4 + 6\ell_1\ell_2\right)\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^5 \\ &+ \left(-42\ell_1^5 - 28\ell_1^2\ell_2\right)\cdot \\ &\qquad\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^6 \\ &+ \left(132\ell_1^6 + 120\ell_1^3\ell_2 + 4\ell_2^2\right)\cdot \\ &\qquad\left(\xi + x + \ell_1(\xi^2+x^2) + \ell_2(\xi^4+x^4) + \ell_3(\xi^8+x^8)\right)^7 \\ &+ O(x,\xi)^8\big].\end{aligned}$$

Expanding, and substituting the Hazewinkel generators, we have

$$
\begin{aligned}
a_0 &= \xi + O(\xi)^8 \\
a_1 &= 1 - v_1\xi + v_1^2\xi^2 - 2v_1^3\xi^3 \\
&\quad + 3v_1^4\xi^4 - 4v_1^5\xi^5 \\
&\quad + v_1^6\xi^6 + O(\xi)^7 \\
&\equiv 1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7 \mod \langle 2\rangle\xi \\
a_2 &= v_1^2\xi - 4v_1^3\xi^2 + 10v_1^4\xi^3 - 21v_1^5\xi^4 \\
&\quad + 43v_1^6\xi^5 + O(\xi)^6 \\
&\equiv v_1^2\xi + v_1^5\xi^4 + O(\xi)^6 \mod \langle 2\rangle\xi.
\end{aligned}
$$

Substituting into the formula for MC_2, we have (modulo v_2)

$$
\begin{aligned}
MC_2(\xi) =& 6a_1^2 - a_0a_2 - 3v_1a_0a_1 \\
&\equiv 6\left(1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7\right)^2 \\
&\quad - 3\left(\xi + O(\xi)^8\right)\left(v_1^2\xi + v_1^5\xi^4 + O(\xi)^6\right) \\
&\quad - 3v_1\left(\xi + O(\xi)^8\right)\left(1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7\right) \\
&\qquad \mod \langle 2\rangle\xi \\
&= 6 + 9v_1\xi + 12v_1^4\xi^4 + 18v_1^5\xi^5 + 21v_1^6\xi^6 + O(\xi)^7 \mod \langle 2\rangle\xi.
\end{aligned}
$$

Note that, although a_2 is accurate only modulo ξ^6, the product a_0a_2 is accurate modulo ξ^7 and hence MC_2 is accurate modulo ξ^7. Since the lowest-order term is $3 \cdot 2$, we subtract $3 \cdot \langle 2\rangle\xi$ to give

$$MC_2(\xi) \equiv 12v_1\xi - 6v_1^2\xi^2 + v_1^3\xi^3 - 66v_1^4\xi^4 + 270v_1^5\xi^5 - 879v_1^6\xi^6 + O(\xi)^7 \mod \langle 2\rangle\xi.$$

Continuing to reduce in this way gives the following:

$$MC_2(\xi) \equiv \ v_1^6\xi^6 + O(\xi)^7 \qquad \mod \langle 2\rangle\xi.$$

Since the lowest-order term of the right-hand side is non-zero mod 2, the entire expression is non-zero in $BP^*[\![\xi]\!]/\langle 2\rangle\xi$.

6.3. Results at $p = 2$.

$$
\begin{aligned}
\langle 2\rangle\xi = {} & 2 - \xi v_1 + 2\xi^2 v_1^2 + \xi^3\left(-8v_1^3 - 7v_2\right) + \xi^4\left(26v_1^4 + 30v_1v_2\right) \\
& + \xi^5\left(-84v_1^5 - 111v_1^2v_2\right) + \xi^6\left(300v_1^6 + 502v_1^3v_2 + 112v_2^2\right) \\
& + \xi^7\left(-1140v_1^7 - 2299v_1^4v_2 - 960v_1v_2^2 - 127v_3\right) \\
& + \xi^8\left(4334v_1^8 + 9958v_1^5v_2 + 5414v_1^2v_2^2 + 766v_1v_3\right) \\
& + \xi^9\left(-16692v_1^9 - 43118v_1^6v_2 - 29579v_1^3v_2^2 - 2380v_2^3 - 3579v_1^2v_3\right) \\
& + \xi^{10}\left(65744v_1^{10} + 189976v_1^7v_2 + 161034v_1^4v_2^2 + 31012v_1v_2^3 + 17770v_1^3v_3 + 5616v_2v_3\right) \\
& + \xi^{11}\left(-262400v_1^{11} - 837637v_1^8v_2 - 838452v_1^5v_2^2 - 240631v_1^2v_2^3 - 86487v_1^4v_3\right. \\
& \qquad \left. -55329v_1v_2v_3\right) \\
& + \xi^{12}\left(1056540v_1^{12} + 3685550v_1^9v_2 + 4232750v_1^6v_2^2 + 1600786v_1^3v_2^3 + 58268v_2^4\right. \\
& \qquad \left. +404198v_1^5v_3 + 363210v_1^2v_2v_3\right) \\
& + \xi^{13}\left(-4292816v_1^{13} - 16254540v_1^{10}v_2 - 21110372v_1^7v_2^2 - 10071369v_1^4v_2^3 - 1022466v_1v_2^4\right. \\
& \qquad \left. -1864478v_1^6v_3 - 2193009v_1^3v_2v_3 - 212440v_2^2v_3\right) \\
& + O(\xi)^{14}
\end{aligned}
$$

$$
\begin{aligned}
MC_1(\xi) \equiv {} & \xi^2v_1^2 + \xi^3v_2 + \xi^4\left(v_1^4 + v_1v_2\right) + \xi^7\left(v_1^7 + v_3\right) + \xi^8\left(v_1^8 + v_1v_3\right) \\
& + \xi^9\left(v_1^9 + v_1^6v_2 + v_1^3v_2^2 + v_2^3 + v_1^2v_3\right) + \xi^{10}\left(v_1^{10} + v_1v_2^3 + v_1^3v_3\right) + \xi^{11}\left(v_1^5v_2^2 + v_1v_2v_3\right) \\
& + \xi^{12}\left(v_1^{12} + v_1^9v_2 + v_1^6v_2^2 + v_1^3v_2^3 + v_2^4 + v_1^5v_3\right) + \xi^{13}v_1^4v_2^3 \\
& + O(\xi)^{14} \mod \langle 2\rangle\xi
\end{aligned}
$$

$$
\begin{aligned}
MC_2(\xi) \equiv {} & \xi^6\left(v_1^6 + v_2^2\right) + \xi^7\left(v_1^7 + v_3\right) + \xi^8\left(v_1^5v_2 + v_1v_3\right) + \xi^9v_2^3 + \xi^{10}\left(v_1^4v_2^2 + v_1v_2^3\right) \\
& + \xi^{11}\left(v_1^5v_2^2 + v_1^2v_2^3 + v_1^4v_3\right) + \xi^{12}\left(v_1^9v_2 + v_1^5v_3\right) + \xi^{13}\left(v_1^{13} + v_1^{10}v_2 + v_1^3v_2v_3\right) \\
& + O(\xi)^{14} \mod \langle 2\rangle\xi
\end{aligned}
$$

$$
\begin{aligned}
MC_3(\xi) \equiv {} & \xi^6v_1^6 + \xi^7\left(v_1^4v_2 + v_1v_2^2\right) + \xi^8\left(v_1^8 + v_1^5v_2 + v_1v_3\right) + \xi^{10}\left(v_1^{10} + v_1^7v_2 + v_1^4v_2^2 + v_1^3v_3 + v_2v_3\right) \\
& + \xi^{11}\left(v_1^{11} + v_1^8v_2 + v_1^4v_3 + v_1v_2v_3\right) + \xi^{12}v_1^3v_2^3 + \xi^{13}\left(v_1^{13} + v_1^3v_2v_3 + v_2^2v_3\right) \\
& + O(\xi)^{14} \mod \langle 2\rangle\xi
\end{aligned}
$$

$$
\begin{aligned}
MC_4(\xi) \equiv {} & \xi^{10}v_1^4v_2^2 + \xi^{11}\left(v_1^{11} + v_1^8v_2 + v_1^5v_2^2 + v_1^4v_3\right) \\
& + \xi^{12}\left(v_1^9v_2 + v_1^3v_2^3 + v_2^4\right) + \xi^{13}\left(v_1^{10}v_2 + v_1^4v_2^3 + v_1^6v_3 + v_1^3v_2v_3 + v_2^2v_3\right) \\
& + O(\xi)^{14} \mod \langle 2\rangle\xi
\end{aligned}
$$

$$
MC_5(\xi) \equiv 0 + O(\xi)^{14} \mod \langle 2\rangle\xi
$$

6.4. **Results at** $p = 3$.

$$\begin{aligned}\langle 3\rangle\xi = 3 &- 8\xi^2 v_1 + 72\xi^4 v_1^2 - 840\xi^6 v_1^3 \\ &+ \xi^8\left(9000v_1^4 - 6560v_2\right) + \xi^{10}\left(-88992v_1^5 + 216504v_1v_2\right) \\ &+ \xi^{12}\left(658776v_1^6 - 5360208v_1^2v_2\right) + \xi^{14}\left(1199088v_1^7 + 119105576v_1^3v_2\right) \\ &+ \xi^{16}\left(-199267992v_1^8 - 2424100032v_1^4v_2 + 129120480v_2^2\right) \\ &+ \xi^{18}\left(5896183992v_1^9 + 45824243688v_1^5v_2 - 8307203592v_1v_2^2\right) \\ &+ \xi^{20}\left(-133449348816v_1^{10} - 807801733088v_1^6v_2 + 336744805688v_1^2v_2^2\right) \\ &+ \xi^{22}\left(2658275605728v_1^{11} + 13162584394728v_1^7v_2 - 11021856839856v_1^3v_2^2\right) \\ &+ \xi^{24}\left(-48579725371464v_1^{12} - 193206868503840v_1^8v_2 + 314960186505360v_1^4v_2^2\right. \\ &\qquad \left.-3670852206240v_2^3\right) \\ &+ O(\xi)^{26}\end{aligned}$$

$$\begin{aligned}MC_2(\xi) \equiv\ & v_1^3\xi^8 + 2v_2\xi^{10} + \left(v_1^5 + v_2v_1\right)\xi^{12} + 2v_1^2v_2\xi^{14} + 2v_1^7\xi^{16} + \left(2v_1^8 + v_2^2\right)\xi^{18} \\ &+ \left(v_2v_1^5 + v_2^2v_1\right)\xi^{20} + \left(2v_1^{10} + 2v_2v_1^6 + v_2^2v_1^2\right)\xi^{22} + \left(v_1^{11} + v_2v_1^7\right)\xi^{24} \\ &+ O(\xi)^{26} \qquad \bmod \langle 3\rangle\xi\end{aligned}$$

$$MC_4(\xi) \equiv 2v_1^9\xi^{22} + 2v_1^{10}\xi^{24} + O(\xi)^{26} \qquad \bmod \langle 3\rangle\xi$$

Note. For $p > 3$, we omit $\langle p\rangle\xi$ and $MC_{(p-1)}$.

6.5. **Results at** $p = 5$.

$$MC_8(\xi) \equiv 3v_1^{16}\xi^{88} + (4v_1^{17} + v_1^{11}v_2)\xi^{92} + (3v_1^{18} + 4v_1^6v_2^2)\xi^{96} + O(\xi^{100}) \quad \bmod \langle 5\rangle\xi$$

6.6. **Results at** $p = 7$.

$$\begin{aligned}MC_{12}(\xi) \equiv\ & 4v_1^{22}\xi^{192} + (4v_1^{23} + 2v_1^{15}v_2)\xi^{198} + (6v_1^{24} + 4v_1^{16}v_2 + 5v_1^8v_2^2)\xi^{204} \\ &+ (5v_1^{25} + 5v_1^{17}v_2 + 4v_1^9v_2^2 + 3v_1v_2^3)\xi^{210} + (2v_1^{18}v_2 + 3v_1^{10}v_2^2 + 4v_1^2v_2^3)\xi^{216} \\ &+ O(\xi^{222}) \quad \bmod \langle 7\rangle\xi\end{aligned}$$

6.7. **Results at** $p = 11$.

$$MC_{20}(\xi) \equiv 9v_1^{34}\xi^{520} + (8v_1^{35} + 6v_1^{23}v_2)\xi^{530} + (7v_1^{36} + v_1^{24}v_2 + 5v_1^{12}v_2^2)\xi^{540} + O(\xi^{550}) \quad \bmod \langle 11\rangle\xi$$

6.8. **Results at** $p = 13$.

$$MC_{24}(\xi) \equiv 11v_1^{40}\xi^{744} + (6v_1^{41} + 6v_1^{27}v_2)\xi^{756} + O(\xi^{768}) \quad \bmod \langle 13\rangle\xi$$

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