DERIVED SMOOTH MANIFOLDS: A SIMPLICIAL APPROACH

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Abstract. Derived smooth manifolds are constructed using the usual homotopy theory of simplicial rings of smooth functions. They are proved to be equivalent to derived smooth manifolds of finite type, constructed using homotopy sheaves of homotopy rings (D. Spivak), thus preserving the classical cobordism ring. This reduction to the usual algebraic homotopy can potentially lead to virtual fundamental classes beyond obstruction theory.

1. Introduction

In [Sp10], Spivak constructed a category of derived manifolds and a notion of cobordism for derived manifolds. Derived manifolds are one of many possible generalizations of smooth manifolds. Approximately, derived manifolds is the smallest simplicial category containing smooth manifolds, their (possibly non-transverse) intersections, and countable unions. Intersections are derived in the sense that they are homotopy pullbacks.

Spivak extended the cobordism relation to his derived category of manifolds and constructed the derived cobordism ring of a manifold. Although derived manifolds include many non-manifolds, the theory is still close enough to the classical situation to obtain a map from the derived cobordism ring to the ordinary cobordism ring and this map is an isomorphism [Sp10, Thm. 2.6].

The category of derived manifolds is constructed with some standard, albeit technologically involved, homotopy theoretic machinery. Although we leave these details to [Sp10] and the body of this paper, the reader should keep in mind that a smooth manifold can be though of as a topological space which admits a nice covering, paired with a (structure) sheaf of rings satisfying certain properties, and that derived manifolds will be similarly defined but with the structure sheaf replaced by a homotopical variant.

In many respects this is a perfectly good definition; we take a suitable definition of smooth manifolds, and derive it in the standard way. This methodology also works for other classes of structured spaces. Although these new categories will have many desirable formal properties, it can be

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difficult to make computations. This situation may be unavoidable in general, but in the case of (derived) smooth manifolds, we have a powerful tool at our disposal: partitions of unity.

In classical differentiable geometry, the existence of partitions of unity allows one to extend local constructions to global ones in a way that is typically impossible in the general theory of sheaves. This is a consequence of manifolds being paracompact, Hausdorff, and equipped with a sheaf of smooth functions (the structure sheaf) that is *soft*. The complete elimination of sheaf theoretic techniques is possible because the softness of structure sheaves on classical manifolds remains true in the derived setting: every derived smooth manifold of finite type[1] (as defined in [Sp10]) is locally weakly equivalent to a derived manifold, whose structure sheaf is a simplicial diagram of soft sheaves.

This ability to express intersections homotopically correctly and functorially, by using simplicial rings, instead of a homotopy sheaf of such rings, potentially allows one to go beyond obstruction theories in working with virtual fundamental classes.

Do you have a references where people use these obstruction theories for constructing virtual fundamental classes?

We will construct a sequence of algebraically defined full simplicial subcategories of the category of derived manifolds. Although these categories are far from essentially surjective we will show in we use C^∞-rings as the basis for everything we do in smooth geometry. Such rings are just algebras over a particular algebraic theory ([La63]), we recall the definition at the beginning of Section 2. The theory of C^∞-rings is well developed (e.g. [MR91], [Du81], [GS03], [Jo11a], and many others).

Simplicial C^∞-rings inherit a simplicial closed model structure from the category of simplicial sets ([Qu67]), and it is this homotopy theory that we use. We recall the definition in Section 3.

In [Sp10] weak equivalences are based on the notion of local weak equivalence. Given a topological space X, one considers the category of sheaves of simplicial C^∞-rings, and defines local weak equivalence to be a map that induces an isomorphism between the corresponding sheaves of homotopy groups.

In contrast to the situation in algebraic geometry, local weak equivalences in differential geometry can be treated in a very simple manner. Using softness of structure sheaves, we prove that every derived manifold of finite type is locally weakly equivalent to a derived manifold, whose presheaves of homotopy groups are already sheaves. This implies that the functor of global sections maps local weak equivalences to weak equivalences, and so we

[1] A derived manifold is of finite type, if it is possible to embed its underlying space into $\mathbb{R}^n$ for some $n \geq 0$.

can go from sheaves to simplicial C^∞-rings without loosing any homotopical structure.

In [Sp10], instead of pre-sheaves of C^∞-rings, pre-sheaves of *h*omotopy C^∞-rings are used, which are not endowed with a point-set C^∞-ring structure, but whose homotopy groups have this structure. The inclusion of strict C^∞ rings into homotopy C^∞ rings is the right adjoint in a Quillen equivalence of model categories ([Ba02], [Be06]). Thus we get the following diagram of adjunctions:

What things are adjoint? Do you mean spaces equipped with a sheaf of C^∞ rings? Probably not, but I'm not sure what you mean here. There is probably an 'op' on the bottom there. I would like to fill this out a bit more. I suspect most of my issues in the introduction would be answered if this diagram and the accompanying explanation was more complete.

(1.1)
$$\begin{array}{ccc} \text{Sheaves of simplicial } C^\infty\text{-rings} & \longleftrightarrow & \text{Spivak's theory} \\ & \overset{\Gamma}{\nwarrow\!\searrow} & \\ & & \text{Simplicial } C^\infty\text{-rings} \end{array}$$

where the categories in the first row come with local weak equivalences, the third category has the usual weak equivalences. Assuming all manifolds are of finite type, each functor maps (local) weak equivalences to (local) weak equivalences. Moreover, all units and counits are (local) weak equivalences.

Going over to simplicial localizations of these categories, we conclude that these adjunctions induce weak equivalences. In [Sp10] derived manifolds are defined by gluing affine derived manifolds, which are just intersections, computed as homotopy limits. Weak equivalences of simplicial localizations, given by (1.1), show that all of this can be done in the category of simplicial C^∞-rings, using the usual closed model structure.

In Section 2 we describe the correspondence between C^∞-spaces and C^∞-rings. Using softness of the structure sheaves, one proves that this is an equivalence of categories. This material is standard, and we provide it mostly to fix the notation.

There is no mention of germ determined, complete, etc.. here.

We extend these results in Section 3 to the simplicial case. We show that the opposite of the standard model category of simplicial C^∞-rings is a model for

I don't think you mean we 'glue' along local weak equivalences. Gluing corresponds to ordinary fibered products.

the category of simplicial C^∞-spaces, where

isn't the category of C^∞ spaces defined here? What do you mean?

gluing is performed by local weak equivalences.

I think it needs to be clearer here, what is the main category that we are comparing everything to

In Section 4 we recall the construction from [Sp10], and (assuming finite type) prove that it provides a model for simplicial C^{∞}-spaces. Thus [Sp10] is reduced, via the category of simplicial C^{∞}-spaces, to the usual construction, involving homotopy colimits in the category of simplicial C^{∞}-rings. This correspondence, restricted to good fibrant replacements on the side of [Sp10], is just the functor of global sections.

We would like to indicate some differences between the approach to derived geometry adopted here, and other places. One can choose the underlying topological spaces to be spectra of the 0-th homotopy group or the 0-th component of the structure sheaf. We (and [Sp10]) use the former, while [CK01],[CK02],[TV05],[TV08] use the latter. Note that it is impossible in the latter approach to define a notion of *Spec* which respects weak equivalences.

In [Jo11b] another approach to derived manifolds is used, where instead of sheaves of simplicial C^{∞}-rings, one uses sheaves of presentations of C^{∞}-rings. We believe that constructions of [Jo11a] can be obtained from ours by truncating simplicial sets from

Perhaps you can explain this to me. Our category has only non-degenerate indecomposables in degrees 0 and 1.

level 2 and up. In particular, this would allow one to reformulate the theory of [Jo11b] in terms of C^{∞}-rings, and their modules. We will pursue this elsewhere.

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2. C^{∞}-rings and C^{∞}-spaces

Let $\mathcal{C}^{\infty}\mathcal{R}$ be the category of product preserving functors

$$A : \mathcal{C}^{\infty} \longrightarrow Set,$$

where $\mathcal{C}^{\infty}$ has $\{\mathbb{R}^n\}_{n\geq 0}$ as objects, and smooth maps as morphisms. Clearly, any such A is determined (up to a unique isomorphism) by the set $A(\mathbb{R})$ and the action of $\{C^{\infty}(\mathbb{R}^n)\}_{n\geq 0}$ on $A(\mathbb{R})$, making it into a *C^{∞}-ring*. We will write A to mean both the functor and the corresponding C^{∞}-ring.

As an example consider a smooth manifold X, by which we will always mean a Hausdorff, second countable space with a smooth finite dimensional euclidean atlas. The set of smooth functions on X is a C^∞-ring, moreover the assignment $X \mapsto C^\infty(X)$ is a full and faithful functor ([MR91, 3.1.4]).

Move definition of ideals to here, recall quotients and that Hadamard makes it work. Explain why the quotient fields are never complex.

A C^∞-ring A is *local*, if A has a unique maximal ideal $\mathfrak{m} \subset A$, and $A/\mathfrak{m} \cong \mathbb{R}$. A typical example of a local C^∞-ring is the ring of germs of smooth functions at the origin of $\mathbb{R}^n$.

A C^∞-ring A is *finitely generated* if A is a quotient of $C^\infty(\mathbb{R}^n)$. Following [MR91], we will denote the full subcategory of $\mathcal{C}^\infty\mathcal{R}$, consisting of finitely generated C^∞-rings, by $\mathcal{L}$.

Note that via an embedding into $\mathbb{R}^n$, the C^∞-ring of smooth functions of any finite dimensional smooth manifold is finitely generated.

would be nice to have notation for generators

For each $a \in A \cong C^\infty(\mathbb{R}^n)/\mathfrak{A}$, let the *localization of* A at $a \neq 0$,

$$A\{a^{-1}\} = C^\infty(\mathbb{R}^{n+1})/(\mathfrak{A}, ax - 1)$$

where x generates the new factor. For any $A \in \mathcal{L}$, *spectrum of* A is the pair $(Sp(A), \mathcal{O}_{Sp(A)})$, where

$$Sp(A) := Hom_{\mathcal{L}}(A, \mathbb{R}),$$

is a topological space with open sets

$$U_a := \{p : A \to \mathbb{R} \mid p(a) \neq 0\}, \tag{2.1}$$

for each $a \in A$, and $\mathcal{O}_{Sp(A)}$ is the sheaf on $Sp(A)$ associated to the presheaf $U_a \mapsto A[a^{-1}]$.

By choosing a presentation $A \cong C^\infty(\mathbb{R}^n)/\mathfrak{A}$, one can identify $Sp(A)$ with the functions on $\mathbb{R}^n$ that vanish on $\mathfrak{A}$. Since every open subset U of $\mathbb{R}^n$ has a smooth function whose support is precisely U, ([MR91, lem. I.1.4]), $Sp(A)$ is a topological subspace of $\mathbb{R}^n$, and every open subset of $Sp(A)$ is of the form (2.1). By comparing germs we see that $\mathcal{O}_{Sp(A)}$ is the pullback of the sheaf of smooth functions on $\mathbb{R}^n$ via this inclusion.

I propose the previous sentence. Or some small change of it, to make it correct as a substitute for the next paragraph

Similarly, there is another description of $\mathcal{O}_{Sp(A)}$. Let $\mathcal{O}_{\mathbb{R}^n}$ be the sheaf of C^∞-functions on $\mathbb{R}^n$, and let $\mathfrak{a} \subseteq \mathcal{O}_{\mathbb{R}^n}$ be subsheaf of

What is the definition of ideal sheaf? Do we really need this?

ideals, defined as follows:

$$f \in \Gamma(U, \mathfrak{a}) \text{ if and only if } \forall p \in U, f_p \in \mathfrak{A}_p.$$

The inclusion $\iota : Sp(A) \subseteq \mathbb{R}^n$ is given by $C^\infty(\mathbb{R}^n) \to A$, it is easy to see that

$$\mathcal{O}_{Sp(A)} \cong \iota^*(\mathcal{O}_{\mathbb{R}^n})/\iota^*(\mathfrak{a}), \tag{2.2}$$

Why is the structure sheaf soft? I can see why this is true provided that the inclusion $Sp(A) \to \mathbb{R}^n$ is closed.

and hence $\mathcal{O}_{Sp(A)}$ is soft, and its stalks are local C^∞-rings.

Recall that a sheaf $\mathcal{O}$ on a space X is *soft* if for all closed subsets $W \subset X$ the restriction map on global sections $\Gamma(X, \mathcal{O}) \to \Gamma(\mathcal{O}, W)$ is surjective.

Recall nice properties of softness

You labeled this (internally) as Classical Definition. Is this definition used elsewhere? Or are we presenting it for the first time. If so some justification may be in order.

The definitions I have found for soft require paracompactness, although I don't see why this has to be part of the definition. We use this condition later, without a condition that we are locally euclidean. It should follow from the embedding result but I guess that will appear later.

Definition 2.1. *A C^∞-space* is a pair $(X, \mathcal{O}_X)$, such that

(1) X is a (paracompact?) Hausdorff topological space,
(2) $\mathcal{O}_X$ is a soft sheaf of finitely generated C^∞-rings on X,
(3) for each point p the stalk at p

$$(\mathcal{O}_X)_p = \mathrm{colim}_{p \in U} O_x(U)$$

is a local C^∞-ring.

A morphism of C^∞-spaces

$$(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair $(\phi, \phi^\sharp)$, where $\phi : X \to Y$ is a continuous map, and $\phi^\sharp : \mathcal{O}_Y \to \phi_*(\mathcal{O}_X)$ is a morphism of sheaves of C^∞-rings.

Note that the corresponding morphisms between stalks are automatically local, since they are morphisms of local $\mathbb{R}$-algebras, that have $\mathbb{R}$ as the residue field. We will denote the category of C^∞-spaces by $\mathbb{G}$.

Note that we require the structure sheaf to be a sheaf of finitely generated C^∞-rings, that is our C^∞-spaces are always *of finite type*, which is equivalent to demanding that our space is embeddable into some $\mathbb{R}^n$. It follows that every C^∞-space is paracompact, second-countable, and Hausdorff.

We will now show that Sp is right adjoint to Γ (cf. [Du81, Thm. 8]). First note that as a subspace of $\mathbb{R}^n$, $Sp(A)$ is clearly Hausdorff, and since the structure sheaf is pulled back from $\mathbb{R}^n$ it is local, if the inclusion of $Sp(A)$ into $\mathbb{R}^n$ is closed then since the closed subsets of $Sp(A)$ come from intersecting with closed subsets of $\mathbb{R}^n$ and the structure sheave on $\mathbb{R}^n$ is soft we see $\mathcal{O}_{Sp(A)}$ is soft.

soft in the general case?

Therefore $(Sp(A), \mathcal{O}_{Sp(A)}) \in \mathbb{G}$, and universal property of localization gives us a functor

$$\mathbf{Sp} : \mathcal{L}^{op} \to \mathbb{G}, \quad A \mapsto (Sp(A), \mathcal{O}_{Sp(A)}).$$

In general, it is not true that $A \cong \Gamma(Sp(A), \mathcal{O}_{Sp(A)})$. Consider the following example: let $A := C^\infty(\mathbb{R}^2)/\mathfrak{A}$, where $f \in \mathfrak{A}$ if f vanishes on some product neighborhood of the y-axis. Then $\Gamma(Sp(A), \mathcal{O}_{Sp(A)}) = C^\infty(\mathbb{R}^2)/\widetilde{\mathfrak{A}}$, where $\widetilde{\mathfrak{A}}$ is the ideal of functions that vanish in some (arbitrary) neighborhood of the y-axis. Clearly $\mathfrak{A} \neq \widetilde{\mathfrak{A}}$.

This example can be generalized. An ideal $\mathfrak{A} \subseteq C^\infty(\mathbb{R}^n)$ is called *germ determined*,[2] if

$$\forall f \in C^\infty(\mathbb{R}^n) - \mathfrak{A}, \exists p \in \mathbb{R}^n \text{ s.t. } f_p \notin \mathfrak{A}_p.$$

A C^∞-ring A is germ determined, if $A \cong C^\infty(\mathbb{R}^n)/\mathfrak{A}$, with $\mathfrak{A}$ being germ determined. We will denote the full subcategory of $\mathcal{L}$, consisting of germ determined C^∞-rings, by $\mathcal{G}$. The inclusion $\mathcal{G} \subset \mathcal{L}$ has a left adjoint, that we will denote by $A \mapsto \widetilde{A}$. Explicitly: $\widetilde{A} \cong C^\infty(\mathbb{R}^n)/\widetilde{\mathfrak{A}}$, where $\widetilde{\mathfrak{A}}$ is the germ determined ideal, generated by $\mathfrak{A}$. Proof of the following proposition is straightforward (use (2.2)).

Proposition 2.2. If A is a finitely generated C^∞ ring then $\Gamma(U_a, \mathcal{O}_{Sp(A)}) \cong \widetilde{A\{a^{-1}\}}$.

Theorem 2.3. Let $(X, \mathcal{O}_X) \in \mathbb{G}$, then $\Gamma(X, \mathcal{O}_X) \in \mathcal{G}$. Moreover,

$$Sp : \mathcal{G}^{op} \rightleftarrows \mathbb{G} : \Gamma$$

is an equivalence of categories.

Proof. Using the results above we see that Sp and Γ restrict to functors between $\mathcal{G}^{op}$ and $\mathbb{G}$. Since $\mathbb{G}$ is a full subcategory of $\mathbb{L}$ and $\mathcal{G}$ is a full subcategory of $\mathcal{L}$, it suffices to prove the adjunction between the larger categories. This result plus the fact that Γ is represented by the C^∞ space $\mathbb{R}$ is precisely [Du81, Thm. 8]. Let $(X, \mathcal{O}_X) \in \mathbb{G}$, using this adjunction and Proposition 2.2, we see that $\Gamma(X, \mathcal{O}_X) \in \mathcal{G}$ and if A is germ determined, $\Gamma(Sp(A), \mathcal{O}_{Sp(A)}) \cong A$.

Let $A := \Gamma(X, \mathcal{O}_X)$. The only thing left to prove is that the canonical map

$$(\iota, \iota^\sharp) : (X, \mathcal{O}_X) \to (Sp(A), \mathcal{O}_{Sp(A)}),$$

is an isomorphism. This map takes a point x to the map $ev_x A \to \mathbb{R}$. First we check surjectivity: Suppose that $p : A \to \mathbb{R} \in Sp(A)$ is not in the image. First we note that p is always surjective and any map of $\mathbb{R}$ algebras lifts to a C^∞ map [MR91].

When p is injective $A \cong \mathbb{R}$ and that X must be a point and $X \cong Sp(\mathbb{R})$. So let $\mathfrak{m}_p$ be the non-trivial kernel of p.

[2]In [Du81] such ideals are called *ideals of local character*, we adopt the terminology from [MR91].

I'm having a hard time seeing this, since the ring is non-noetherian as a commutative algebra-¿

Since $\Gamma(X, \mathcal{O}_X)$ is finitely generated, all of its maximal ideals, having $\mathbb{R}$ as the residue field, are finitely generated, and hence there are $\alpha_1, \ldots, \alpha_n \in \mathfrak{m}_p$ generating $\mathfrak{m}_p$. If there was a point x such that $\ker ev_x \supset \mathfrak{m}$ then the two ideals would be equal and p would be in the image. It follows that $\alpha_1^2 + \ldots + \alpha_n^2 \neq 0$ on X, contradicting the surjectivity of p.

To prove injectivity, we take any two distinct points x and y and find a global section s such that $ev_x(s) \neq ev_y(s)$. Since X is Hausdorff we can find disjoint neighborhoods U_x and U_y of x and y respectively. So there is a section $\Gamma(x, y, \mathcal{O}_X)$ that is nonzero at x and zero at y. Using softness we extend this to obtain our desired global section.

The inverse map takes the open set U_a to the set V_a of $x \in X$ such that $ev_x(a) \neq 0$. By the representability result of Dubuc a corresponds to a continuous map $X \to \mathbb{R}$ and V_a is the support of this map, hence open.

It suffices to check the isomorphism on structure sheaves after taking germs at an arbitrary point p. Since $\mathcal{O}_X$ is soft, $A = \Gamma(X, \mathcal{O}_X) \to (\mathcal{O}_X)_p$ is surjective, and hence $(\mathcal{O}_{Sp(A)})_p \to (\mathcal{O}_X)_p$ is surjective as well. Suppose that $a \in A$, maps to $0 \in (\mathcal{O}_X)_p$, then it a vanishes in an open neighborhood U of p Let $b \in A$, such that $b_p = 1_p \in (\mathcal{O}_{Sp(A)})_p$, $b_q = 0_q \in (\mathcal{O}_{Sp(A)})_q$ $\forall q \in X - U$. Then, clearly, $(ab)_q \mapsto 0_q \in (\mathcal{O}_X)_q$ $\forall q \in X$, i.e., $ab \mapsto 0 \in \Gamma(X, \mathcal{O}_X)$. Therefore $ab = 0 \in A$, and hence $a_p = 0 \in (\mathcal{O}_{Sp(A)})_p$, so $\mathcal{O}_{Sp(A)} \to \mathcal{O}_X$ is an isomorphism. □

From Theorem 2.3 it is clear, that for a finitely generated C^∞-ring, being germ determined is equivalent to being *geometric* in the sense that it is smooth functions on a C^∞-space. The requirement to be finitely generated is essential, since even the free C^∞-ring $C^\infty(\mathbb{R}^S)$, with S infinite, is not isomorphic to $\Gamma(Sp(C^\infty(\mathbb{R}^S)), \mathcal{O}_{Sp(C^\infty(\mathbb{R}^S))})$.

Yet, one has the notion of a germ determined C^∞-ring also in the infinitely generated case ([Bo11]). In general, for A to be germ determined is equivalent to $A \to \Gamma(Sp(A), \mathcal{O}_{Sp(A)})$ being injective.

Why aren't we considering fine sheaves? The locally complete case looks relatively simple in terms of fine sheaves.

Surjectivity is more complicated. If $Sp(A)$ is paracompact, it is enough for A to be *locally complete*,[3] which means for every set of elements $\{a_i\}$ such that only finitely many a_i are non-trivial in any neighborhood of a point, then there is a global section which restricts to the sum of these elements locally. Any finitely generated, germ determined C^∞-ring is complete in this sense, but not always true otherwise.

This definition immediately extends to modules over a C^∞-ring, by which we mean an ordinary module over the underlying commutative ring.

[3]In [Jo11a] this is called being *complete with respect to locally finite sums.*

We will denote the category of such A-modules by $\overline{Mod}_A$. Since sheaves of modules over soft sheaves of rings are themselves soft, we have the following conclusion.

Do you want germ determined here (otherwise can we recover A from a non-germ determined A? Quasicoherent sheaves?

Proposition 2.4. Let A be a finitely generated C^∞-ring. The functor of global sections defines an equivalence between the category of sheaves of $\mathcal{O}_{Sp(A)}$-modules and $\overline{Mod}_A$.

3. Simplicial C^∞-rings and simplicial C^∞-spaces

The category of simplicial C^∞-rings $s\mathcal{C}^\infty\mathcal{R}$, the category of simplicial objects in C^∞ rings.

Standard techniques show that $s\mathcal{C}^\infty\mathcal{R}$ is a simplicial closed model category, where *weak equivalences* and *fibrations* are those morphisms that define respectively weak equivalences and fibrations between the underlying simplicial sets [Qu67]. The forgetful functors to simplicial $\mathbb{R}$-algebras and $\mathbb{R}$-modules reflect weak equivalences and homotopy groups can be calculated in these underlying categories.

In particular, since each simplicial C^∞-ring is a simplicial $\mathbb{R}$-module, it is enough to calculate homotopy groups at 0, and,

$$\pi_k(A_\bullet, 0) \cong H_k(\mathcal{M}(A_\bullet)),$$

where $\mathcal{M}(A_\bullet) = (\bigoplus A_k, \Sigma(-1)^i\partial_{k+1,i})$ is the *Moore complex* of $A_\bullet$.

Since π_k preserves products $\pi_k A_\bullet$ is a C^∞ ring and an $\mathbb{R}$-module over $\pi_0 A_\bullet$.

Let X be a topological space, and let $\mathcal{O}_{\bullet,X}$ be a sheaf of simplicial C^∞-rings on X. Let $\pi_k(\mathcal{O}_{\bullet,X})$ be the sheaf associated to the presheaf

$$U \mapsto \pi_k(\Gamma(U, \mathcal{O}_{\bullet,X})$$

By the comments above, $\pi_0(\mathcal{O}_{\bullet,X})$ is a sheaf of C^∞-rings, and $\{\pi_k(\mathcal{O}_{\bullet,X})\}_{k\geq 1}$ are sheaves of $\pi_0(\mathcal{O}_{\bullet,X})$-modules.

Perhaps the name sC^∞ space is better, simplicial C^∞-space makes me think of a simplicial object in C^∞ spaces.

Definition 3.1. *A simplicial C^∞-space* is a pair $(X, \mathcal{O}_{\bullet,X})$, where X is a topological space, and $\mathcal{O}_{\bullet,X}$ is a sheaf of simplicial C^∞-rings on X, and $(X, \pi_0(\mathcal{O}_{\bullet,X}))$ is a C^∞-space in the sense of Definition 2.1.

A morphism $(X, \mathcal{O}_{\bullet,X}) \to (Y, \mathcal{O}_{\bullet,Y})$ is given by a pair $(\phi, \phi^\sharp_\bullet)$, where $\phi : X \to Y$ is a continuous map, and $\phi^\sharp_\bullet := \{\phi^\sharp_k : \mathcal{O}_{k,Y} \to \phi_*(\mathcal{O}_{k,X})\}_{k\geq 0}$ is a morphism of sheaves of simplicial C^∞-rings. We will denote the category of simplicial C^∞-spaces by $s\mathbb{G}$.

3.1. **Local weak equivalences.** Given $(X,\mathcal{O}_{\bullet,X}) \in s\mathbb{G}$, we have a C^∞-space $(X,\pi_0(\mathcal{O}_{\bullet,X}))$ and a sequence $\{\pi_k(\mathcal{O}_{\bullet,X})\}_{k\geq 1}$ of sheaves of modules on it. We will denote this C^∞-space, together with the sheaves of modules, by $\pi_\bullet(X,\mathcal{O}_{\bullet,X})$. Every morphism

$$(\phi,\phi^\sharp_\bullet) : (X,\mathcal{O}_{\bullet,X}) \to (Y,\mathcal{O}_{\bullet,Y})$$

in $s\mathbb{G}$ induces a morphism

$$\pi_\bullet(\phi,\phi^\sharp_\bullet) : \pi_\bullet(X,\mathcal{O}_{\bullet,X}) \to \pi_\bullet(Y,\mathcal{O}_{\bullet,Y}).$$

We call $(\phi,\phi^\sharp_\bullet)$ a *local weak equivalence*, if $\pi_\bullet(\phi,\phi^\sharp_\bullet)$ is an isomorphism.

I dislike this name for two reasons. A 'local weak equivalence' is a standard term, it is usually just a weak equivalence of simplicial sheaves (over a fixed base), which is a piece of the above definition. You are also including a homeomorphism in the data. Second, this is not a local property (because of the homeomorphism condition).

Let $s\mathcal{L} \subset s\mathcal{C}^\infty\mathcal{R}$ be the full subcategory, consisting of $A_\bullet$, such that $\pi_0(A_\bullet)$ is a finitely generated C^∞-ring. There is an obvious functor

$$\Gamma : s\mathbb{G} \to s\mathcal{L}^{op}, \quad (X,\mathcal{O}_{\bullet,X}) \mapsto \Gamma(X,\mathcal{O}_{\bullet,X}).$$

In general Γ does not map local weak equivalences in $s\mathbb{G}$ to weak equivalences in $s\mathcal{L}^{op}$. It does so for a particular kind of simplicial C^∞-spaces, that we are going to consider next.

Definition 3.2. *A localized simplicial C^∞-space* is a C^∞-space

This name is also a bit strange, because these spaces aren't 'localized.' Since this is exactly the simplicial C^∞ spaces that are locally ringed in the standard sense. Why not call them locally ringed simplicial C^∞ spaces.

$(X,\mathcal{O}_{\bullet,X})$, where the stalks of $\mathcal{O}_{0,X}$ are local C^∞-rings. We will denote by $\overline{s\mathbb{G}} \subset s\mathbb{G}$ the full subcategory, consisting of localized simplicial C^∞-spaces.

Proposition 3.3. Let $(X,\mathcal{O}_{\bullet,X}) \in \overline{s\mathbb{G}}$, then $\mathcal{O}_{0,X}$ is a soft sheaf.

Proof. Since $(X,\pi_0(\mathcal{O}_{\bullet,X}))$ is a C^∞-space, X is paracompact.

Note that this was not in the definition of C^∞ space. If you required it there, I think there are some more conceptual proofs of the results you stated.

By [Go60, Thm. II.3.7.2], to prove that $\mathcal{O}_{0,X}$ is soft, it is enough to show that it is locally soft, i.e., $\forall p \in X$ there is a neighborhood $U \ni p$, such that $\forall V_1, V_2 \subseteq U$ closed, with $V_1 \cap V_2 = \emptyset$, there is $\alpha \in \Gamma(U,\mathcal{O}_{0,X})$, such that $\alpha_{V_1} = 0$, $\alpha_{V_2} = 1$.

Let $p \in X$, and let $\{f_i\}_{1\leq i\leq n}$ be a set of generators of $\Gamma(X,\pi_0(\mathcal{O}_{\bullet,X}))$ as a C^∞-ring. Since $(\mathcal{O}_{0,X})_p \to (\pi_0(\mathcal{O}_{\bullet,X}))_p$ is surjective, there is a neighborhood $W \ni p$, such that $\{f_i|_W\}_{i=1}^n$ are in the image of $\Gamma(W,\mathcal{O}_{0,X}) \to \Gamma(W,\pi_0(\mathcal{O}_{\bullet,X}))$. Choose a closed neighborhood $U \ni p$, such that $U \subseteq W$, we claim that U satisfies the conditions of theorem II.3.7.2 in [Go60].

Indeed, since $\pi_0(\mathcal{O}_{\bullet,X})$ is soft, $\Gamma(X, \pi_0(\mathcal{O}_{\bullet,X})) \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective, and hence $\Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is generated by the images of $\{f_i\}_{i=1}^n$. Therefore $\Gamma(U, \mathcal{O}_{0,X}) \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective. We can choose a finitely generated C^∞-subring $A \subseteq \Gamma(U, \mathcal{O}_{0,X})$, such that $A \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ is surjective. We claim that $U \cong Sp(A)$ as topological spaces.

Suppose not, surjectivity of $A \to \Gamma(U, \pi_0(\mathcal{O}_{\bullet,X}))$ implies that $U \subseteq Sp(A)$ as a closed subspace, hence $Sp(A) - U$ is non-empty and open. Therefore, $\exists a \in A$, such that $a \neq 0$, $a_U = 0$, when considered as a section of $\mathcal{O}_{Sp(A)}$. Since A is finitely generated, $\exists b \in A$, such that $ab = 0$ and $b(q) = 1$ $\forall q \in U$. Since $(X, \mathcal{O}_{\bullet,X})$ is localized, b is invertible in $\Gamma(U, \mathcal{O}_{0,X})$, and therefore $a = 0$ as a section of $\mathcal{O}_{0,X}$, contradiction to A being a C^∞-subring. So $U \cong Sp(A)$.

From $U \cong Sp(A)$ and A being finitely generated, we conclude that $\forall V_1, V_2 \subseteq U$ closed, such that $V_1 \cap V_2 = \emptyset$, there is $\alpha \in A$, such that $\alpha_{V_1} = 0$, $\alpha_{V_2} = 1$, where we consider α as a section of $\mathcal{O}_{Sp(A)}$. Since $(X, \mathcal{O}_{\bullet,X})$ is localized, α has the same properties, when considered as a section of $\mathcal{O}_{0,X}$. and $\beta\alpha = 0$. Now consider β as a section of $\mathcal{O}_{0,X}$. From $\beta(a) = 1$ we conclude that β_q is invertible, from $\beta\alpha = 0$ we conclude that $\alpha_q = 0$. In the case of V_2 we do the same with $\alpha - 1$. □

Let $(X, \mathcal{O}_{\bullet,X}) \in \overline{s\mathbb{G}}$. Since $\mathcal{O}_{0,X}$ is soft, all sheaves of boundaries in $\mathcal{M}(\mathcal{O}_{\bullet,X})$ are soft as well, and hence (see e.g. [Go60], theorem II.3.5.2)

$$\forall k \geq 0 \quad \Gamma(X, \pi_k(\mathcal{O}_{\bullet,X})) \cong \pi_k(\Gamma(X, \mathcal{O}_{\bullet,X})). \tag{3.1}$$

This immediately implies the following proposition.

Proposition 3.4. The functor of global sections $\Gamma : \overline{s\mathbb{G}} \to s\mathcal{L}^{op}$ maps local weak equivalences to weak equivalences.

It might appear that localized simplicial C^∞-spaces are very special. However, the following proposition shows that every simplicial C^∞-space can be *localized*, such that the result is locally weak equivalent to the original space.

Proposition 3.5. The inclusion $\overline{s\mathbb{G}} \subset s\mathbb{G}$ has a right adjoint. Unit of this adjunction is an isomorphism, and counit consists of local weak equivalences.

Proof. Let $(X, \mathcal{O}_{\bullet,X}) \in s\mathbb{G}$, and let $p \in X$. The stalk $(\mathcal{O}_{0,X})_p$ does not have to be local, but $(\pi_0(\mathcal{O}_{\bullet,X}))_p$ is local. Therefore, since $(\pi_0(\mathcal{O}_{\bullet,X}))_p$ is a quotient of $(\mathcal{O}_{0,X})_p$, we get a distinguished

$$\mathfrak{p} : (\mathcal{O}_{0,X})_p \longrightarrow \mathbb{R}.$$

Let $(\mathcal{O}_{0,X})_\mathfrak{p}$ be the localization of $(\mathcal{O}_{0,X})_p$ at $\mathfrak{p}$, i.e., it is obtained by universally inverting every $f_p \in (\mathcal{O}_{0,X})_p$, whose value at $\mathfrak{p}$ is not 0. As usual with C^∞-rings, the natural map

$$(\mathcal{O}_{0,X})_p \to (\mathcal{O}_{0,X})_\mathfrak{p} \tag{3.2}$$

is surjective. We will denote kernel of (3.2) by $\mathfrak{t}_p$. For an open $U \subseteq X$, we will call a section $\alpha \in \Gamma(U, \mathcal{O}_{0,X})$ *trivial*, if $\forall p \in U$ $\alpha_p \in \mathfrak{t}_p$. It is clear that all trivial sections together comprise a sheaf of ideals $\mathfrak{t} \subset \mathcal{O}_{0,X}$. It is easy to check that $\forall p \in X$ $\mathfrak{t}_p$ is exactly the set of germs of $\mathfrak{t}$ at p.

We define

$$\overline{\mathcal{O}}_{\bullet,X} := \mathcal{O}_{\bullet,X}/\mathfrak{t}\mathcal{O}_{\bullet,X},$$

where the right hand side is taken in the category of sheaves of simplicial C^∞-rings, i.e., we first divide, and then sheafify. Proposition 3.9 gives the middle isomorphism in

$$\forall p \in X, \quad (\pi_0(\overline{\mathcal{O}}_{\bullet,X}))_p \cong \pi_0((\overline{\mathcal{O}}_{\bullet,X})_p) \cong \pi_0((\mathcal{O}_{\bullet,X})_p) \cong (\pi_0(\mathcal{O}_{\bullet,X}))_p,$$

and hence $(X, \overline{\mathcal{O}}_{\bullet,X}) \in s\mathbb{G}$. Since stalks of $\overline{\mathcal{O}}_{\bullet,X}$ are local C^∞-rings, we have that in fact $(X, \overline{\mathcal{O}}_{\bullet,X}) \in \overline{s\mathbb{G}}$. It is clear that $(X, \mathcal{O}_{\bullet,X}) \mapsto (X, \overline{\mathcal{O}}_{\bullet,X})$ extends to a functor $s\mathbb{G} \to \overline{s\mathbb{G}}$, and this functor is right adjoint to $\overline{s\mathbb{G}} \subset s\mathbb{G}$.

If $(X, \mathcal{O}_{\bullet,X})$ is localized, there are no non-zero trivial sections, and hence unit of the adjunction is an isomorphism. Using Proposition 3.9 again, we conclude that the map $\mathcal{O}_{\bullet,X} \to \overline{\mathcal{O}}_{\bullet,X}$ is a weak equivalence stalk-wise, i.e., counit of the adjunction consists of local weak equivalences. $\square$

Let $\mathbf{\Gamma} : s\mathbb{G} \to s\mathcal{L}^{op}$ be the composition of the localization functor $s\mathbb{G} \to \overline{s\mathbb{G}}$ and the functor of global sections $\Gamma : \overline{s\mathbb{G}} \to s\mathcal{L}^{op}$. Propositions 3.4 and 3.5 imply that $\mathbf{\Gamma}$ maps local weak equivalences to weak equivalences. Our next objective is to define a functor, going in the opposite direction.

3.2. **Spectrum of a simplicial C^∞-ring.** Let $A_\bullet \in s\mathcal{L}$, i.e., $A_\bullet$ is a simplicial C^∞-ring, such that $\pi_0(A_\bullet)$ is finitely generated as a C^∞-ring. We define

$$Sp(A_\bullet) := Sp(\pi_0(A_\bullet)).$$

Let $U \subseteq Sp(A_\bullet)$ be open. For any $p \in U$ we have an evaluation

$$\mathfrak{p} : \ A_0 \to \pi_0(A_\bullet) \xrightarrow{p} \mathbb{R}.$$

Let $\mathcal{U} := \{a \in A_0 \text{ s.t. } \forall p \in U, \mathfrak{p}(a) \neq 0\}$, we define $A_\bullet\{\mathcal{U}^{-1}\}$ to be the simplicial C^∞-ring, obtained by universally inverting every $a \in \mathcal{U}$, and its degeneracies in $A_{\geq 1}$. $U = Sp(A_\bullet)$, we will write $\overline{A}_\bullet$ instead of $A_\bullet\{\mathcal{U}^{-1}\}$, and call it the *localization of* $A_\bullet$.

It is clear that $U \mapsto A_\bullet\{\mathcal{U}^{-1}\}$ is a presheaf of simplicial C^∞-rings on $Sp(A_\bullet)$. Let $\mathcal{O}_{\bullet,Sp(A_\bullet)}$ be the associated sheaf.

Proposition 3.6. Let $(Sp(A_\bullet), \mathcal{O}_{\bullet,Sp(A_\bullet)})$ be defined as above. Then

$$\pi_0(\mathcal{O}_{\bullet,Sp(A_\bullet)}) \cong \mathcal{O}_{Sp(\pi_0(A_\bullet))}.$$

Proof. Recall that $\mathcal{O}_{Sp(\pi_0(A_\bullet))}$ is sheafification of the presheaf

$$U \mapsto \pi_0(A_\bullet)\{[\mathcal{U}]^{-1}\}, \quad U \subseteq Sp(\pi_0(A_\bullet)),$$

where $[\mathcal{U}] \subseteq \pi_0(A_\bullet)$ consists of those elements, that do not vanish at any $p \in U$, i.e., $\mathcal{U} \subseteq A_0$ is the pre-image of $[\mathcal{U}]$. Therefore we have a canonical $A_0\{\mathcal{U}^{-1}\} \to \pi_0(A_\bullet)\{[\mathcal{U}]^{-1}\}$, and it obviously factors through $\pi_0(A_\bullet\{\mathcal{U}^{-1}\})$. the co-equalizer of $A_1\{\mathcal{U}^{-1}\} \rightrightarrows A_0\{\mathcal{U}^{-1}\}$, but composed with $A_0\{\mathcal{U}^{-1}\} \to$

$\pi_0(A_\bullet)\{[\mathcal{U}^{-1}]\}$, these two maps become equal, since these compositions are given by the universal property of localization, applied to $A_1 \rightrightarrows A_0 \to \pi_0(A_\bullet)$. Using universal properties of localization and sheafification, we arrive at

$$\pi_0(\mathcal{O}_{\bullet,Sp(A_\bullet)}) \to \mathcal{O}_{Sp(\pi_0(A_\bullet))}. \tag{3.3}$$

To prove that (3.3) is an isomorphism, we need to look at the stalks. Let $p \in Sp(A_\bullet)$, then $(\mathcal{O}_{Sp(\pi_0(A_\bullet))})_p \cong \pi_0(A_\bullet)/\mathfrak{m}_p^g$, $(\mathcal{O}_{\bullet,Sp(A_\bullet)})_p \cong A_\bullet/\mathfrak{m}_\mathfrak{p}^g A_\bullet$, where $\mathfrak{m}_p^g \subset \pi_0(A_\bullet)$, $\mathfrak{m}_\mathfrak{p}^g \subset A_0$ consist of elements that have 0 germs at respectively p and $\mathfrak{p}$ (see Proposition 3.8). Let $\mathfrak{k}$ be the kernel of $A_0 \to \pi_0(A_\bullet)$. It is easy to see that the left hand side of (3.3) is $A_0/\mathfrak{k}+\mathfrak{m}_\mathfrak{p}^g$, while the right hand side is $(A_0/\mathfrak{k})/\mathfrak{m}_p^g$. It is straightforward to see that $A_0/\mathfrak{k}+\mathfrak{m}_\mathfrak{p}^g = (A_0/\mathfrak{k})/\mathfrak{m}_p^g$. $\square$

The previous proposition implies that $(Sp(A_\bullet), \mathcal{O}_{\bullet,A_\bullet}) \in s\mathbb{G}$. Since construction of $\mathcal{O}_{\bullet,Sp(A_\bullet)}$ involves only inverting elements, it is clearly functorial. Moreover, it is obvious that stalks of $\mathcal{O}_{0,Sp(A_\bullet)}$ are local C^∞-rings, so we have a functor

$$\mathbf{Sp} : s\mathcal{L}^{op} \to \overline{s\mathbb{G}}, \quad A_\bullet \mapsto (Sp(A_\bullet), \mathcal{O}_{\bullet,Sp(A_\bullet)}).$$

In general, it is not true that $A_\bullet$ is weakly equivalent to $\Gamma(Sp(A_\bullet), \mathcal{O}_{\bullet,Sp(A_\bullet)})$. Even in the simple case when $A_\bullet$ is a constant simplicial diagram if $A_\bullet$ is not germ determined, then $A \ncong \Gamma(Sp(A), \mathcal{O}_{Sp(A)})$.

A simplicial C^∞-ring $A_\bullet$ is *geometric*, if it is homotopically finitely generated and germ determined, i.e., if

(1) $\pi_0(A_\bullet)$ is a finitely generated, germ determined C^∞-ring,
(2) $\forall k \geq 1$, $\pi_k(A_\bullet)$ is a germ determined, locally complete $\pi_0(A_\bullet)$-module (see

change to exact reference

Section 2 for definition of local completeness).

Let $s\mathcal{G}$ be the full subcategory of $s\mathcal{L}$, consisting of geometric simplicial C^∞-rings. If $(X, \mathcal{O}_{\bullet,X}) \in \overline{s\mathbb{G}}$, from (3.1) we know, that for any $k \geq 0$ $\pi_k(\Gamma(X, \mathcal{O}_{\bullet,X})) \cong \Gamma(X, \pi_k(\mathcal{O}_{\bullet,X}))$, and hence $\pi_0(\Gamma(X, \mathcal{O}_{\bullet,X}))$ is finitely generated, germ determined, and $\pi_k(\Gamma(X, \mathcal{O}_{\bullet,X}))$ are all germ determined, locally complete $\pi_0(\Gamma(X, \mathcal{O}_{\bullet,X}))$-modules. Therefore, the functor of global sections Γ maps $\overline{s\mathbb{G}}$ to $s\mathcal{G}^{op}$.

Proposition 3.7. The functor of global sections $\Gamma : \overline{s\mathbb{G}} \to s\mathcal{G}^{op}$ is left adjoint to $\mathbf{Sp} : s\mathcal{G}^{op} \to \overline{s\mathbb{G}}$. Unit of this adjunction is an isomorphism, the counit consists of weak equivalences.

Proof. Let $(X, \mathcal{O}_{\bullet,X}) \in \overline{s\mathbb{G}}$, $A_\bullet \in s\mathcal{G}$. Any morphism $f_\bullet : A_\bullet \to \Gamma(X, \mathcal{O}_{\bullet,X})$ induces $\pi_0(f_\bullet) : \pi_0(A_\bullet) \to \pi_0(\Gamma(X, \mathcal{O}_{\bullet,X})) \cong \Gamma(X, \pi_0(\mathcal{O}_{\bullet,X}))$, and hence a continuous map $\phi : X \to Sp(A_\bullet)$. Since stalks of $\mathcal{O}_{0,X}$ are local, universal property of localization implies that $f_\bullet$ defines $\phi_\bullet^\sharp : \mathcal{O}_{\bullet,Sp(A_\bullet)} \to \phi_*(\mathcal{O}_{\bullet,X})$.

Let $\overline{A}_\bullet$ be obtained by universally inverting every $a \in A_0$, such that $\mathfrak{p}(a) \neq 0$ for any $p \in Sp(\pi_0(A_\bullet))$. From Proposition 3.8 we know that

$A_\bullet \to \overline{A}_\bullet$ is surjective, and hence, using universal property of sheafification and proceeding as in the proof of Theorem 2.3, we conclude that $f_\bullet \mapsto (\phi, \phi^\sharp_\bullet)$ is a bijective correspondence. It is clearly functorial, i.e., **Sp** is right adjoint to Γ.

Suppose that $A_\bullet = \Gamma(X, \mathcal{O}_{\bullet,X})$. Since $\pi_0(A_\bullet) \cong \Gamma(X, \pi_0(\mathcal{O}_{\bullet,X}))$, obviously $X \cong Sp(A_\bullet)$ as topological spaces. Then, since $\mathcal{O}_{0,X}$ is soft, and its stalks are local, one proves that $\mathcal{O}_{\bullet,Sp(A_\bullet)} \cong \mathcal{O}_{\bullet,X}$ as in Theorem 2.3. So the unit of the adjunction is indeed an isomorphism.

Now we compare $A_\bullet$ and $\Gamma(Sp(A_\bullet), \mathcal{O}_{Sp(A_\bullet)})$. By Proposition 3.9 we know that $A_\bullet \to \overline{A}_\bullet$ is a weak equivalence. It remains to show that sheafification preserves this homotopy type, i.e., that

$$\gamma : \overline{A}_\bullet \to \Gamma(Sp(A_\bullet), \mathcal{O}_{Sp(A_\bullet)})$$

is a weak equivalence. We can assume that $A_\bullet = \overline{A}_\bullet$. Let $\alpha \in \mathcal{M}(A_\bullet)$ be a cycle, such that $\gamma(\alpha)$ is a boundary. This implies that $\forall p \in Sp(A_\bullet)$, $\gamma(\alpha)_p$ is a boundary in $(\mathcal{O}_{\bullet,Sp(A_\bullet)})_p$. From Proposition 3.8 we conclude that $(\mathcal{O}_{\bullet,Sp(A_\bullet)})_p \cong A_\bullet/\mathfrak{m}^g_\mathfrak{p} A_\bullet$, therefore we can construct $\beta \in A_0$, such that $\alpha\beta$ is homologous to 0, yet $\mathfrak{p}(\beta) \neq 0$. This implies that $\alpha \mapsto 0$ in $H_\bullet(\mathcal{M}(A_\bullet))_p$. Since this happens for every p, and we assume $\pi_k(A_\bullet)$ to be germ determined $\forall k \geq 0$, α has to be a boundary itself.

Now let $\xi \in \Gamma(Sp(A_\bullet), \mathcal{O}_{\bullet,Sp(A_\bullet)})$ be a cycle. Let $p \in Sp(A_\bullet)$, then $\xi_p \in (\mathcal{O}_{\bullet,Sp(A_\bullet)})_p \cong A_\bullet/\mathfrak{m}^g_\mathfrak{p} A_\bullet$ is a cycle. It is easy to see that for any $\alpha \in A_0$, whose support is contained in a small open set around $\mathfrak{p}$, $\alpha\xi_p$ extends to a cycle in $A_\bullet$. Therefore, using partition of unity, we can find a family of cycles $\{\beta_i\} \subseteq A_\bullet$, such that $\{\gamma(\beta_i)\}$ is locally finite and

$$\forall p \in Sp(A_\bullet), \quad \sum_i (\beta_i)_\mathfrak{p} = \xi_p.i$$

Since $\forall k \geq 0$, $\pi_k(A_\bullet)$ is locally complete, the corresponding family of homology classes $\{[\beta_i]\}$ adds to one class $[\beta] \in H_\bullet(\mathcal{M}(A_\bullet))$. It is clear that $\gamma(\beta)$ is homologous to ξ. □

We have started with the category $s\mathbb{G}$, and we were interested in the simplicial localization of $s\mathbb{G}$ with respect to local weak equivalences. We have constructed two adjunctions:

$$\overline{s\mathbb{G}} \rightleftarrows s\mathcal{G}^{op}, \quad \overline{s\mathbb{G}} \rightleftarrows s\mathbb{G}.$$

In each case the unit and counit consist of local weak equivalences (in the case of $s\mathbb{G}$, $\overline{s\mathbb{G}}$), or weak equivalences (in the case of $s\mathcal{G}$). Therefore simplicial localization of $s\mathcal{G}^{op}$, with respect to weak equivalences, is weakly equivalent to simplicial localization of $s\mathbb{G}$ with respect to local weak equivalences.

We finish this section with technical results, that were used in some of the propositions above.

Proposition 3.8. Let $A \in \mathcal{C}^{\infty}\mathcal{R}$, and let $\mathfrak{I} \subset A$ be an ideal, such that $A/\mathfrak{I}$ is finitely generated. Let $V \subseteq Sp(A/\mathfrak{I})$ be closed, and let

$$\Lambda := \{a \in A \text{ s.t. } \forall p \in V \ \mathfrak{p}(a) \neq 0\}.$$[4]

Let A_V be obtained by inverting every $a \in \Lambda$. Let $S \subseteq A$ be a set of generators. Then

$$A_V \cong C^{\infty}(\mathbb{R}^S)/\mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}},$$

where $\mathfrak{A} \subset C^{\infty}(\mathbb{R}^S)$ is the kernel of $C^{\infty}(\mathbb{R}^S) \to A$, and $\mathfrak{m}^g_{\mathcal{V}} \subset C^{\infty}(\mathbb{R}^S)$ consists of functions, that have 0 germ at $C^{\infty}(\mathbb{R}^S) \to A \to A/\mathfrak{I} \xrightarrow{p} \mathbb{R}$ for any $p \in V$.

Proof. Since $\mathfrak{A} \subseteq \mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}}$, there is a canonical $\phi : A \to C^{\infty}(\mathbb{R}^S)/\mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}}$. First we prove that $\forall a \in \Lambda$ $\phi(a)$ is invertible. Let $\widetilde{a} \in C^{\infty}(\mathbb{R}^S)$ be any pre-image of a. Since $A/\mathfrak{I}$ is finitely generated, we can choose a finite $F \subseteq S$, such that $C^{\infty}(\mathbb{R}^F) \hookrightarrow C^{\infty}(\mathbb{R}^S) \to A/\mathfrak{I}$ is surjective, and $\widetilde{a} \in C^{\infty}(\mathbb{R}^F)$. Let $\mathcal{V}_F$ be the image of V in $\mathbb{R}^F$. Since $\widetilde{a}(p) \neq 0$ $\forall p \in \mathcal{V}$, using partition of unity on $\mathbb{R}^F$, we can find $\widetilde{b} \in C^{\infty}(\mathbb{R}^F)$, such that $\widetilde{a}\widetilde{b} - 1 = 0$ in a neighborhood of $\mathcal{V}_F$. This implies that $\widetilde{a}\widetilde{b} - 1$, considered as an element of $C^{\infty}(\mathbb{R}^S)$, has 0 germ in a neighborhood of every $p \in \mathcal{V}$, i.e. $\widetilde{a}\widetilde{b} - 1 \in \mathfrak{m}^g_{\mathcal{V}}$, and hence $\phi(a)$ is invertible in $C^{\infty}(\mathbb{R}^S)/\mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}}$.

Since ϕ is obviously surjective, for any $A \to B$, that inverts every $a \in \Lambda$, there is at most one factorization through ϕ. To prove that at least one such factorization exists, we construct $\chi : C^{\infty}(\mathbb{R}^S)/\mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}} \to A_V$, such that the following diagram is commutative.

$$\begin{array}{ccc} & A & \\ {\scriptstyle \psi_0} \swarrow & & \searrow {\scriptstyle \phi} \\ A_V & \xleftarrow[\chi]{} & C^{\infty}(\mathbb{R}^S)/\mathfrak{A} + \mathfrak{m}^g_{\mathcal{V}} \end{array}$$

To construct this χ, it is enough to show that kernel of $C^{\infty}(\mathbb{R}^S) \to A_V$ contains $\mathfrak{m}^g_{\mathcal{V}}$. Let $\widetilde{a} \in \mathfrak{m}^g_{\mathcal{V}}$, and choose a finite $F \subseteq S$, such that $\widetilde{a} \in C^{\infty}(\mathbb{R}^F)$, and $C^{\infty}(\mathbb{R}^F) \to A/\mathfrak{I}$ is surjective, let $\mathcal{V}_F$ be the image of V in $\mathbb{R}^F$. By assumption, there is an open $\mathcal{U}_F \supseteq \mathcal{V}_F$, such that $\widetilde{a} = 0$ on $\mathcal{U}_F$. Using paracompactness of $\mathbb{R}^F$, we can find $\widetilde{b} \in C^{\infty}(\mathbb{R}^F)$, such that $\widetilde{b}(p) \neq 0$ $\forall p \in \mathcal{V}_F$ and $supp(\widetilde{b}) \subseteq \mathcal{U}_F$. This means that $\widetilde{a}\widetilde{b} = 0$, yet the image of $\widetilde{b}$ in A_V is invertible. Hence $\widetilde{a} \mapsto 0 \in A_V$. □

Proposition 3.9. Let $A_\bullet$ be a simplicial C^{∞}-ring, such that $\pi_0(A_\bullet)$ is finitely generated and germ determined. Let $\overline{A}_\bullet$ be obtained by universally inverting every $a \in A_0$, such that $\mathfrak{p}(a) \neq 0$, $\forall p \in Sp(\pi_0(A_\bullet))$. Then the natural map $\phi_\bullet : A_\bullet \to \overline{A}_\bullet$ is a weak equivalence.

[4]Here $\mathfrak{p}$ is the composition $A \to A/\mathfrak{I} \xrightarrow{p} \mathbb{R}$.

Proof. From Proposition 3.8 we know that $\overline{A}_0 \cong A_0/\mathfrak{m}$, for some ideal $\mathfrak{m} \subseteq A_0$. It is easy to see that $\overline{A}_\bullet \cong A_\bullet/\mathfrak{m}A_\bullet$.

Clearly $\mathfrak{m}\mathcal{M}(A_\bullet)$ is a subcomplex, and hence to prove the proposition it is enough to show that $\mathfrak{m}\mathcal{M}(A_\bullet)$ is acyclic. Let $k \geq 0$, and let $\beta = \sum\limits_{1 \leq i \leq n} b_i \beta_i$, where $b_i \in \mathfrak{m}$ and $\beta_i \in A_k$. Since $\pi_0(A_\bullet)$ is finitely generated, and there are only finitely many of b_i's, one can use partition of unity to find $a \in A_0$, such that $a\beta = 0$, yet $\mathfrak{p}(a) \neq 0 \; \forall \mathfrak{p} \in \mathcal{V}$. The latter implies that the image of a in $\pi_0(A_\bullet)$ is invertible ($\pi_0(A_\bullet)$ is assumed to be germ determined), and hence, if β is a cycle, it is necessarily a trivial one.

Now assume there is $\alpha \in A_{k+1}$, such that $d\alpha = \beta$. Using partition of unity again, we can find $b \in \mathfrak{m}$, such that $bb_i = b_i$ for all $1 \leq i \leq n$. □

4. Spivak's construction

In this section we show that construction of derived manifolds, done in [Sp10], can be equivalently performed in the category $s\mathcal{L}$ of simplicial C^∞-rings $A_\bullet$, such that $\pi_0(A_\bullet)$ is finitely generated.

The meeting point of Spivak's construction and the usual homotopy theory of simplicial C^∞-rings is the category $s\mathbb{G}$ of simplicial C^∞-spaces, defined in Section 3. We start with recalling (and somewhat reformulating) the constructions of [Sp10].

Let **CG** be the category of compactly generated Hausdorff spaces. We define a category **RS** as follows:

(1) objects are pairs $(X, O_{\bullet,X})$, where $X \in \mathbf{CG}$ and $O_{\bullet,X}$ is a functor
$$Open(X)^{op} \times \mathcal{C}^\infty \to SSet,$$
(2) morphisms are pairs $(\phi, \phi^\sharp)$, where $\phi : X \to Y$ is a continuous map, and $\phi^\sharp$ is a natural transformation

$$\begin{array}{ccc} Open(Y)^{op} \times \mathcal{C}^\infty & \xrightarrow{\phi^{-1} \times Id} & Open(X)^{op} \times \mathcal{C}^\infty \\ & \searrow \quad \xrightarrow{\phi^\sharp} \quad \swarrow & \\ & SSet & \end{array}$$

For any fixed $X \in \mathbf{CG}$, we will denote by $\mathbf{RS}(X) \subset \mathbf{RS}$ the full subcategory consisting of pre-sheaves on X. We equip each $\mathbf{RS}(X)$ with the injective closed model structure, where $O_{\bullet,X} \to O'_{\bullet,X}$ is a weak equivalence or cofibration, if $O_{\bullet,X}(U, \mathbb{R}^n) \to O'_{\bullet,X}(U, \mathbb{R}^n)$ is respectively a weak equivalence or cofibration of simplicial sets, $\forall U \in Open(X)$, $\forall n \geq 0$.

Since we have not required $O_{\bullet,X}(U, -) : \mathcal{C}^\infty \to SSet$ to be product preserving, the natural maps

$$O_{\bullet,X}(U, \mathbb{R}^{m+n}) \to O_{\bullet,X}(U, \mathbb{R}^m) \times O_{\bullet,X}(U, \mathbb{R}^n) \tag{4.1}$$

are not required to be isomorphisms, and not even weak equivalences, i.e., $O_{\bullet,X}$ is not necessarily a pre-sheaf of C^∞-rings. Similarly, having a hyper-cover $\{U_i\}$ of U, the natural map

$$O_{\bullet,X}(U,\mathbb{R}^k) \to holim(O_{\bullet,X}(U_i,\mathbb{R}^k)) \tag{4.2}$$

does not have to be a weak equivalence, i.e., $O_{\bullet,X}(-,\mathbb{R}^k)$ is not necessarily a homotopy sheaf of simplicial sets.

Since $\mathbf{RS}(X)$ is a left proper, cellular, simplicial closed model category, one can perform a left localization of $\mathbf{RS}(X)$ with respect to (4.1) and (4.2) [Hi09]. The result is a left proper, simplicial closed model category, that we will denote by $\mathbf{Shv(X)}$. Moreover, any continuous map $\phi : X \to Y$ induces a Quillen adjunction

$$\phi^* : \mathbf{Shv}(Y) \rightleftarrows \mathbf{Shv}(X) : \phi_*.$$

Homotopy sheaves of homotopy simplicial C^∞-rings on X are the fibrant objects in $\mathbf{Shv}(X)$. Every $O_{\bullet,X} \in \mathbf{Shv}(X)$ is cofibrant, and we will denote by $\mathcal{O}_{\bullet,X}$ a chosen functorial fibrant replacement. For any $U \in Open(X)$, $\mathcal{O}_{\bullet,X}(U,-)$ is a homotopy C^∞-ring, i.e., (4.1) is a weak equivalence $\forall m,n \geq 0$, and therefore we have a well defined sheaf of C^∞-rings $\pi_0(\mathcal{O}_{\bullet,X})$, and a sequence of sheaves of $\pi_0(\mathcal{O}_{\bullet,X})$-modules $\{\pi_k(\mathcal{O}_{\bullet,X})\}_{k\geq 1}$.

A weak equivalence in $\mathbf{RS}$ is a morphism $(\phi,\phi^\sharp) : (X, O_{\bullet,X}) \to (Y, O_{\bullet,Y})$, such that ϕ is a homeomorphism, and $\phi^\sharp : O_{\bullet,Y} \to \phi_*(O_{\bullet,X})$ is a weak equivalence in $\mathbf{Shv}(Y)$. Equivalently, we can demand that $\mathcal{O}_{\bullet,Y} \to \phi_*(\mathcal{O}_{\bullet,X})$ is a local weak equivalence, i.e., it induces isomorphisms

$$\pi_k(\mathcal{O}_{\bullet,Y}) \to \phi_*(\pi_k(\mathcal{O}_{\bullet,X})), \quad \forall k \geq 0.$$

We will denote by $\underline{\mathbf{RS}}$ the simplicial localization of $\mathbf{RS}$ with respect to these weak equivalences. Presence of simplicial closed model structure on each $\mathbf{Shv}(X)$ makes computing $\underline{\mathbf{RS}}$ easier than usual.

Proposition 4.1. The category $\mathbf{RS}$, together with the subcategory of weak equivalences, admits a homotopy calculus of fractions.

Proof. What exactly are you proving here?

Let $(\phi,\phi^\sharp) : (X, O_{\bullet,X}) \to (Y, O_{\bullet,Y})$ be a weak equivalence in $\mathbf{RS}$, we will say that $(\phi,\phi^\sharp)$ is a trivial cofibration or trivial fibration, if correspondingly $\phi^\sharp$ is a trivial fibration or a trivial cofibration. Using the closed model structure on $\mathbf{Shv}(Y)$, it is obvious that every weak equivalence in $\mathbf{RS}$ can be written as a composition of a trivial cofibration, followed by a trivial fibration.

Consider diagrams

$$
\begin{array}{ccc}
(X, O_{\bullet,X}) & \xrightarrow{(\alpha,\alpha^\sharp)} & (Y, O_{\bullet,Y}) \\
{\scriptstyle (\phi,\phi^\sharp)}\downarrow & & \\
(X', O_{\bullet,X'}) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & (Z, O_{\bullet,Z}) \\
 & & \downarrow{\scriptstyle (\psi,\psi^\sharp)} \\
(W, O_{\bullet,W}) & \xrightarrow[(\beta,\beta^\sharp)]{} & (Z', O_{\bullet,Z'})
\end{array}
\tag{4.3}
$$

where $(\phi, \phi^\sharp)$ is a trivial cofibration, and $(\psi, \psi^\sharp)$ is a trivial fibration. It is easy to see that

$$(Y, O_{\bullet,Y} \coprod_{\alpha_*(O_{\bullet,X})} (\alpha\phi^{-1})_*(O_{\bullet,X'})), \quad (W, O_{\bullet,W} \prod_{\beta^*(O_{\bullet,Z'})} (\psi^{-1}\beta)^*(O_{\bullet,Z}))$$

are the pushout and pullback respectively of (4.3). Since right Quillen functors preserve trivial fibrations, and left Quillen functors

Which Quillen functors?

preserve trivial cofibrations, it is clear that pushout of $(\phi, \phi^\sharp)$ is a trivial cofibration, and pullback of $(\psi, \psi^\sharp)$ is a trivial fibration. Using the 2-out-of-3 property, we see that, if in addition $(\alpha, \alpha^\sharp)$, $(\beta, \beta^\sharp)$ are weak equivalences, their pushout and pullback are weak equivalences as well. Use [DK80a, Prop. 8.2]. □

The same argument shows that any subcategory of **RS**, defined by putting conditions on the sheaves of homotopy groups,

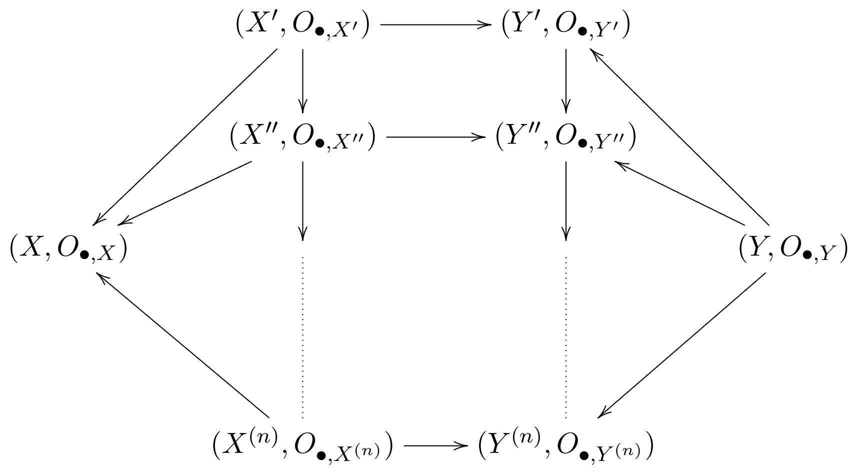
????????? There are weak equivalences in **RS** that are not weak equivalences, even locally. These weak equivalences are not isomorphisms of homotopy groups. I don't see why any such subcategory would be closed under these operations. I don't see why a subcategory would admit functorial factorizations.

also admits a homotopy calculus of fractions. We are interested in the following two subcategories:

- let **LRS** $\subset$ **RS** be the full subcategory, consisting of pairs $(X, O_{\bullet,X})$, such that the stalks of $\pi_0(\mathcal{O}_{\bullet,X})$ are local C^∞-rings,
- let $\mathbf{LRS}_{fgs} \subset$ **LRS** be the full subcategory of pairs $(X, O_{\bullet,X})$, such that $\pi_0(\mathcal{O}_{\bullet,X})$ is a soft sheaf of finitely generated C^∞-rings.

As with **RS**, we will denote by $\underline{\mathbf{LRS}}$, $\underline{\mathbf{LRS}}_{fgs}$ the simplicial localizations with respect to local weak equivalences. Since **RS** admits a homotopy calculus of fractions, we know that $Hom_{\underline{\mathbf{RS}}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y})$ is weakly equivalent to simplicial set of hammocks of the following form:

(4.4)



where vertical arrows, and arrows going to the left are weak equivalences. The same is true for **LRS** and $\mathbf{LRS}_{fgs}$. Since **LRS**, $\mathbf{LRS}_{fgs}$ are full subcategories of **RS**, defined by a condition on weak equivalence classes, we immediately have the following result.

Proposition 4.2. The inclusions

$$\underline{\mathbf{LRS}}_{fgs} \subset \underline{\mathbf{LRS}} \subset \underline{\mathbf{RS}}$$

induce weak equivalences on the spaces of morphisms.

So far we have used only presence of a closed model structure on each $\mathbf{Shv}(X)$. Now we will use these closed model structures are simplicial. For any $(X, O_{\bullet,X})$, $(Y, O_{\bullet,Y}) \in \mathbf{RS}$ we have a simplicial set

$$\coprod_{\phi \in Hom_{\mathbf{CG}}(X,Y)} \coprod_{k \geq 0} Hom_{\mathbf{Shv}(Y)}(O_{\bullet,Y} \otimes \Delta[k], \phi_*(\mathcal{O}_{\bullet,X})),$$

which we will denote by $\underline{Hom}_{\mathbf{RS}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y}))$.

I believe the following result is immediate from standard techniques

Proposition 4.3. For any $(X, O_{\bullet,X})$, $(Y, O_{\bullet,Y)} \in \mathbf{RS}$ there is a weak equivalence of simplicial sets:

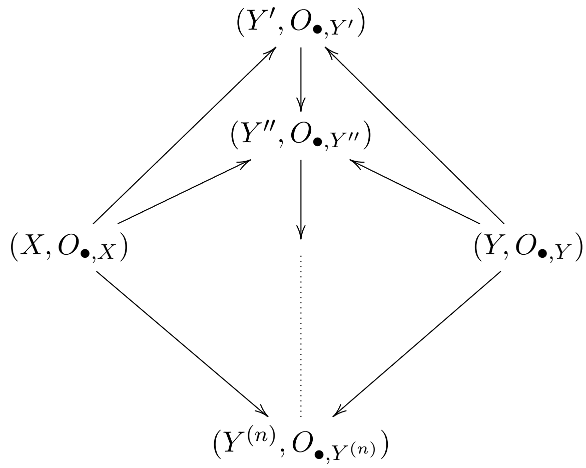
$$\underline{Hom}_{\mathbf{RS}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y})) \simeq Hom_{\underline{\mathbf{RS}}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y})).$$

Proof. Recall from the proof of Proposition 4.1, that $(\phi, \phi^\sharp) : (X, O_{\bullet,X}) \to (Y, O_{\bullet,Y})$ is a trivial fibration, if ϕ is a homeomorphism, and $\phi^\sharp : O_{\bullet,Y} \to \phi^*(O_{\bullet,X})$ is a trivial cofibration. Since cofibrations in each $\mathbf{Shv}(X)$ are just injective maps, it is easy to see that closing trivial fibrations in **RS** with respect to the 2-out-of-3 property, produces all weak equivalences. Hence

$\underline{\mathbf{RS}}$ can be computed as simplicial localization of $\mathbf{RS}$ with respect to trivial fibrations.

Proceeding as in the proof of Proposition 4.1, one sees that $\mathbf{RS}$, together with trivial fibrations, admits a calculus of homotopy right fractions ([DK80a], proposition 8.1). Therefore, $Hom_{\underline{\mathbf{RS}}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y})$ is weakly equivalent to simplicial set of hammocks of the following form:

(4.5)



where vertical arrows, and arrows going to the left are trivial fibrations. Moreover, since nerves of equivalent categories are weakly equivalent, we can assume that $Y^{(k)} = Y \ \forall k \geq 1$.

It is easy to see that in each such hammock, every path from X to Y has the same underlying continuous map $\phi : X \to Y$, and hence we have a decomposition of simplicial sets

$$Hom_{\underline{\mathbf{RS}}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y})) = \coprod_{\phi} Hom_{\underline{\mathbf{RS}}}((X, O_{\bullet,X}), (Y, O_{\bullet,Y}))_{\phi}.$$

Fix a ϕ, using functorial fibrant replacement in each $\mathbf{Shv}(X)$, we can assume that all pre-sheaves in (4.5) are fibrant. Pushing everything forward to Y, we see (4.5) becomes a hammock between $(Y, \phi_*(\mathcal{O}_{\bullet,X}))$ and $(Y, \mathcal{O}_{\bullet,Y})$ in $\mathbf{Shv}(Y)$, and this correspondence is bijective, i.e. we have

$$Hom_{\underline{\mathbf{RS}}}((X, \mathcal{O}_{\bullet,X}), (Y, \mathcal{O}_{\bullet,Y})) \simeq Hom_{\underline{\mathbf{Shv}(Y)}}(\mathcal{O}_{\bullet,Y}, \phi_*(\mathcal{O}_{\bullet,X})),$$

where $\underline{\mathbf{Shv}(\mathbf{Y})}$ is the simplicial localization of $\mathbf{Shv}(Y)$ with respect to trivial cofibrations. Finally, since $\mathcal{O}_{\bullet,Y}$ is cofibrant, and $\phi_*(\mathcal{O}_{\bullet,X})$ is fibrant, we have

$$Hom_{\underline{\mathbf{Shv}(Y)}}(\mathcal{O}_{\bullet,Y}, \phi_*(\mathcal{O}_{\bullet,X})) \simeq \coprod_{k \geq 0} Hom_{\mathbf{Shv}(Y)}(\mathcal{O}_{\bullet,Y} \otimes \Delta[k], \phi_*(\mathcal{O}_{\bullet,X}))$$

Here we use [DK80b, Cor 4.7, Prop. 5.2]. ☐

In [Sp10] $\mathbf{LRS}$ is equipped with simplicial structure, which we will denote by $\widehat{\mathbf{LRS}}$. We have seen now that $\widehat{\mathbf{LRS}}$ is weakly equivalent to $\underline{\mathbf{LRS}}$, that we have constructed here. Therefore, all constructions involving homotopy limits (e.g. derived manifolds), that one can perform in $\widehat{\mathbf{LRS}}$, can be equivalently performed in $\underline{\mathbf{LRS}}$.

Recall ([Sp10]) that *an affine derived manifold* is a homotopy limit (in $\widehat{\mathbf{LRS}}$) of a diagram

(4.6)
$$\begin{array}{ccc} & & \mathbb{R}^0 \\ & & \downarrow \\ \mathbb{R}^m & \longrightarrow & \mathbb{R}^n \end{array}$$

Since $\underline{\mathbf{LRS}} \simeq \widehat{\mathbf{LRS}}$, we can use $\underline{\mathbf{LRS}}$ instead. Moreover, if $(X, \mathcal{O}_{\bullet,X})$ is the homotopy pullback of (4.6), then $\pi_0(\mathcal{O}_{\bullet,X})$ is a soft sheaf of finitely generated C^∞-rings. Therefore affine derived manifolds lie in the full subcategory $\underline{\mathbf{LRS}}_{fgs} \subset \underline{\mathbf{LRS}}$.

Arbitrary derived manifolds in [Sp10] are defined by gluing affine ones. So if we restrict to derived manifolds of finite type, i.e., $(X, \mathcal{O}_{\bullet,X})$ such that $\pi_0(\mathcal{O}_{\bullet,X})$ is a sheaf of finitely generated C^∞-rings, we are still inside $\underline{\mathbf{LRS}}_{fgs}$.

I do not understand any of the arguments that follow.

There is an obvious (full) inclusion

$$s\mathbb{G} \subset \mathbf{LRS}_{fgs}. \tag{4.7}$$

We claim that (4.7) induces a weak equivalence of simplicial localizations (with respect to local weak equivalences). Indeed, any functor $\mathcal{C}^\infty \to SSet$ can be rectified to a simplicial C^∞-ring, i.e., it can be changed into a product

You have to be careful here as to what you mean by homotopy type here. It is not an isomorphism of homotopy groups. Can this rectification be done respecting descent with respect to hypercovers?

preserving functor. Moreover, this process is functorial, and it preserves homotopy type ([Be06]).

Let $< s\mathbb{G} > \subset \mathbf{LRS}$ be the full subcategory, consisting of $s\mathbb{G}$ and all pairs $(X, O_{\bullet,X})$, such that $O_{\bullet,X}$ is weakly product preserving and cofibrant in the

undefined, do you mean projective model structure on simplicial presheaves? The simplicial localization won't depend on which model structure you choose, only on the weak equivalences.

projective closed model structure on $\mathbf{RS}(X)$. Then $s\mathbb{G} \subseteq < s\mathbb{G} >$ induces a weak equivalence between simplicial localizations with respect to local weak equivalences [Lu09, Lem. 5.5.9.9]. On the other hand, using a cofibrant replacement functor $\mathbf{LRS} \to \mathbf{LRS}$ (with respect to the projective model structures on $\mathbf{RS}(X)$), we conclude that $< s\mathbb{G} > \subset \mathbf{LRS}_{fgs}$ also induces a weak equivalences between simplicial localizations.

What results? How do we know this? What exactly are the homotopy limits in these categories? Once you've simplicially localized simplicial C^∞ spaces I don't think I understand what the mapping spaces are anymore (because it is not clear that the objects we started with satisfied descent with respect to hypercovers)

Altogether, using results of Section 3, we now know that taking homotopy limit of (4.6) in $\widehat{\mathbf{LRS}}$ is equivalent to doing the same in $s\mathcal{G}^{op}$, or equivalently in $s\mathcal{L}^{op}$, which has the usual (projective) closed model structure. Therefore:

Theorem 4.4. The simplicial category of derived manifolds of finite type, as defined in [Sp10], is weakly equivalent to the full subcategory of $s\mathcal{G}^{op}$, consisting of objects, that locally are homotopy pullbacks (computed as homotopy pushouts in $s\mathcal{L}$) of (4.6).

Definition 4.5. Given a smooth map $f : \mathbb{R}^{\ltimes} \to \mathbb{R}^m$ let R_f be the homotopy pushout

Theorem 4.6. Every derived manifold is of the form $Sp(A)$ where

$$A = eq\left[\prod_{i\in I} R_{f_i} \rightrightarrows \prod_{j\in J} R_{f_j}\right]$$

References

[Ba02] B.Badzioch. *Algebraic theories in homotopy theory.* Annals of Mathematics, 155, pages 895-913 (2002).

[Be06] J.Bergner. *Rigidification of algebras over multi-sorted theories.* Algebr. Geom. Topol. 6, pages 1925-1955 (2006).

[Bo11] D.Borisov *Topological characterization of various types of C^∞-rings.* arXiv:1108.5885v2 [math.DG] (2011).

[CK01] I.Ciocan-Fontanine, M.Kapranov. *Derived Quot schemes.* Ann. Scient. Éc. Norm. Sup., 4^e série, t. 34, pages 403-440 (2001).

[CK02] I.Ciocan-Fontanine, M.Kapranov. *Derived Hilbert schemes.* Journal of the American Mathematical Society, Vol. 15, Num. 4, pages 787-815 (2002).

[DHI04] D.Dugger, S.Hollander, D.C.Isaksen. *Hypercovers and simplicial presheaves.* Math. Proc. Cambridge Philos. Soc. 136, no. 1, pages 9-51 (2004).

[DK80a] W.G.Dwyer, D.M.Kan. *Calculating simplicial localizations.* Journal of Pure and Applied Algebra 18, pages 17-35 (1980).

[DK80b] W.G.Dwyer, D.M.Kan. *Function complexes in homotopical algebra.* Topology, vol. 19 pages 427-440 (1980).

[Du81] E.J. Dubuc. *C^∞-schemes.* American Journal of Mathematics, Vol. 103, No. 4, pp. 683-690 (1981).

[Go60] R. Godement. *Topologie algébrique et théorie des faisceaux.* viii+283 pages, Hermann (1960).

[GS03] J.A.N. González, J.B.S. de Salas. *C^∞-differentiable spaces.* Lecture Notes in Mathematics 1824, XIII+188 pages, Springer (2003).

[Hi09] Ph.Hirschhorn. *Model categories and their localizations.* 457 pages, AMS (2009).

[Ja87] J.F.Jardine. *Simplicial presheaves.* J. Pure Appl. Algebra 47, pages 35-87 (1987).

[Jo11a] D.Joyce. *Algebraic geometry over C^∞-rings.* math.AG/1001.0023v2

[Jo11b] D.Joyce. *D-manifolds and d-orbifolds: a theory of derived differential geometry.* Preprint, available at http://people.maths.ox.ac.uk/∼joyce/dmanifolds.html

[La63] F.W.Lawvere. *Functorial semantics of algebraic theories.* Proc. Nat. Acad. Sci. U.S.A. 50, pp. 869-872 (1963).

[Lu09] J.Lurie. *Higher Topos theory.* XIX+931 pages, Princeton University Press (2009).

[MR91] I.Moerdijk, G.E.Reyes. *Models for smooth infinitesimal analysis.* X+399 pages, Springer (1991).

[Qu67] D.G. Quillen. *Homotopical algebra.* Lecture Notes in Math. 43, Springer (1967).

[Sp10] D.I.Spivak. *Derived smooth manifolds.* Duke Mathematical Journal, Vol. 153, No. 1, pp. 55-128 (2010).

[TV05] B.Toën, G.Vezzosi. *Homotopical algebraic geometry I: Topos theory.* Adv. in Math. 193, pages 257-372 (2005).

[TV08] B.Toën, G.Vezzosi. *Homotopical algebraic geometry II: geometric stacks and applications.* Mem. Amer. Math. Soc. no. 902, x+224 pages (2008).

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