$H_\infty \neq E_\infty$

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Abstract. We give an example of a spectrum with an H_∞ structure which does not rigidify to an E_3 structure. It follows that not every H_∞ ring spectrum comes from an underlying E_∞ ring spectrum. After comparing definitions, we obtain this example by applying Σ^∞_+ to the counterexample to the transfer conjecture constructed by Kraines and Lada.

1. Introduction

In recent years there has been a renewed interest in the study of E_∞ ring spectra and their strictly commutative analogues, commutative S-algebras. Such spectra are equipped with a well-behaved theory of power operations. In the hands of an expert, this structure provides formidable computational tools which can be used to deduce a number of surprising results (for some examples see [BMMS86, Ch. 2]).

Such operations determine and are determined by an H_∞ ring structure, the analogue of an E_∞ ring structure in the stable *homotopy* category. The theory of power operations is sufficiently rich that one might conjecture that every H_∞ ring spectrum is obtained by taking an E_∞ ring spectrum and then passing to the homotopy category.

This turns out to be a stable analogue of the transfer conjecture. A conjectural equivalence between the homotopy category of infinite loop spaces and a subcategory of the homotopy category of based spaces whose objects admit certain transfer homomorphisms (see [KL79] for a more complete description).

Kraines and Lada demonstrate the falsehood of the transfer conjecture by contructing an explicit counterexample. In their paper, Kraines and Lada define the notion of an $L(n)$ space. When $n = 2$, this is a space equipped with transfer homomorphisms. They also make use of the following implications

$$X \text{ is an infinite loop space} \implies X \text{ is an } E_\infty \text{ space} \implies X \text{ is an } L(\infty) \text{ space.}$$

Theorem 1.1 ([KL79]). *Let s be a generator of $\mathrm{Prim} H^{30}(BU; \mathbb{Z}_{(2)})$. Define KL by the following fibration sequence:*

$$KL \xrightarrow{i} BU_{(2)} \xrightarrow{4s} K(\mathbb{Z}_{(2)}, 30).$$

Then i is a map of $L(2)$ spaces, but the $L(2)$ structure on KL does not lift to an E_3 structure. In particular, KL does not admit an E_∞ structure compatible with this $L(2)$ structure.

After some translation we will prove the following theorem, which provides an example of an H_∞ ring spectrum whose H_∞ structure does not arise by forgetting an E_∞ structure.

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Theorem 1.2. *The map*

$$\Sigma^\infty_+ KL \xrightarrow{\Sigma^\infty_+ i} \Sigma^\infty_+ BU_{(2)}$$

is a map of H_∞ ring spectra, but the H_∞ ring structure on $\Sigma^\infty_+ KL$ does not lift to an E_3 structure. In particular, $\Sigma^\infty_+ KL$ does not admit a compatible E_∞ ring structure.

To prove this we will show that Σ^∞_+ takes $L(2)$ spaces to H_∞ ring spectra and takes infinite loop spaces to E_∞ ring spectra. This comparison is deduced immediately from some of the results in [May09].

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2. $L(n)$ SPACES AND SPECTRA

Notation 2.1. *Let $\mathscr{L}$ be the linear isometries operad. We will abuse notation and let L to denote the associated reduced monad on pointed spaces with Cartesian products, spaces under S^0 with smash products, and spectra under S^0 with smash products.*

In particular:

- L is an endofunctor on pointed spaces satisfying
$$LY = \coprod_{n \geq 0} \mathscr{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$
where $\sim$ represents the obvious base point identifications.
- L is an endofunctor on spaces under S^0 satisfying
$$LY = \coprod_{n \geq 0} \mathscr{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$
where $\sim$ represents the obvious unit map identifications.
- L is an endofunctor on the Lewis-May-Steinberger category of spectra (see [LMS86]) under S^0 satisfying
$$LE = \bigvee_{n \geq 0} \mathscr{L}(n) \ltimes_{\Sigma_n} E^{\wedge n}/(\sim),$$
where $\sim$ represents the obvious unit map identifications (see [EKMM97, 4.9,6.1]).

We justify this abuse of notation with the following lemma:

Lemma 2.2 ([May09, 4.8, p. 1027])**.** *We have the following chain of homeomorphisms natural in based spaces*[1] *X*

$$\begin{aligned} \Sigma^\infty_+ LX &\equiv \Sigma^\infty (LX)_+ \\ &\cong \Sigma^\infty L(X_+) \\ &\cong L\Sigma^\infty X_+ \\ &\equiv L\Sigma^\infty_+ X. \end{aligned}$$

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[1] It is helpful to think of this basepoint as a multiplicative unit.

For simplicity, for the remainder of this paper we will assume all spaces are non-degenerately based.

Recall that the category of L-algebras in group-like pointed spaces is equivalent to the category of infinite loop spaces. The following definition provides a categorical filtration between spaces and homotopy coherent L-algebras (which are weakly equivalent to L-algebras).

Definition 2.3. *A based space X is $L(n)$ if one can construct the $n-1$ skeleton (in the simplicial direction) of the augmented simplicial space*

$$B(L, L, X) \xrightarrow{\mu} X,$$

such that the canonical map

$$X \to LX \hookrightarrow B(L, L, X)$$

is a section of μ.

Remark 2.4. Despite the similarity in notation, we remind the reader that the *property* of being $L(n)$ has nothing to do with the *space* $\mathscr{L}(n)$.

Remark 2.5. Note that our definition of a $L(n)$ space is different from that of a Q_n space used in [KL79]. Kraines and Lada restrict to the case when X is connected, in which case L could be replaced with $Q = \Omega^\infty\Sigma^\infty$. In this respect, our definition is more general.

Our definition differs from that of Kraines and Lada in another way. Their definition of a Q_n space is a cubical analogue of the above definition, while maps of Q_n spaces are defined simplicially. Such a definition requires one to continually translate between these two worlds. We take this opportunity to propose the above alternative definition which is simpler to manipulate and can be easily adapted to any reasonable category of algebras over an operad.

Restricting to connected spaces, one can probably relate the two definitions using the Quillen equivalence between simplicial and cubical sets [Jar06, Cis06]. In any case, we only require these notions to coincide when X is connected and $n \leq 2$, in which case the equivalence is immediate.

We illustrate our definition with a sequence of examples (for more detailed exposition and proofs see [KL79] or [CLM76, V]).

Example 2.6.

1. By definition, every space is a $L(0)$ space.
2. A space X is $L(1)$, if the canonical map $X \to LX$ admits a retraction μ.
3. Let μ_L denote the structure map $L^2 \to L$. A space X is $L(2)$, if it is $L(1)$ and we have a specified homotopy $I \times L^2X \to X$ between $\mu\mu_L$ and $\mu(\mu)$. In other words, X is a strictly unital L-algebra in the homotopy category of pointed spaces.
4. A space X is $L(\infty)$ if and only if it is a strong homotopy retract of a L-algebra. If the components of X form a group under the induced multiplication, then X is $L(\infty)$ if and only if it has the homotopy type of an infinite loop space.

There is an obvious analogue of the above definition with based spaces replaced by spectra over S^0 and L replaced by L. Since the category of L-algebras in spectra over S^0 is isomorphic to the cateogory E_∞ ring spectra [May09, 6.2], we obtain an analogous categorical filtration between spectra under S^0 and E_∞ ring spectra.

Applying this equivalence to the definition of $L(n)$ spectra, we see that the definition of an $L(2)$ spectrum is precisely the definition of a strictly unital H_∞ ring spectrum [BMMS86].

The following proposition provides the necessary comparison to prove Theorem 1.2.

Proposition 2.7.

1. *If X is an $L(2)$ space then $\Sigma^\infty_+ X$ is a strictly unital H_∞ ring spectrum.*
2. *If X is an $L(2)$ space such that the H_∞ ring structure on $\Sigma^\infty_+ X$ rigidifies to an E_∞ structure. Then the $L(2)$ structure on X extends to a $L(\infty)$ structure.*

Proof. The first two parts are obvious from Lemma 2.2 and the comments above. For (2), the essential point is the stable maps defining an L structure on $\Sigma^\infty_+ X$ must come from the unstable $L(2)$ structure on X, which implies we are in the image of the functor Σ^∞_+ from L-algebras in spaces to L-algebras in spectra in Lemma 2.2. □

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