DERIVED INDUCTION AND RESTRICTION THEORY

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Abstract. Let G be a finite group. To any family $\mathscr{F}$ of subgroups of G, we associate a thick $\otimes$-ideal $\mathscr{F}^{\mathrm{Nil}}$ of the category of G-spectra with the property that every G-spectrum in $\mathscr{F}^{\mathrm{Nil}}$ (which we call $\mathscr{F}$-nilpotent) can be reconstructed from its underlying H-spectra as H varies over $\mathscr{F}$. A similar result holds for calculating G-equivariant homotopy classes of maps into such spectra via an appropriate homotopy limit spectral sequence. In general, the condition $E \in \mathscr{F}^{\mathrm{Nil}}$ implies strong collapse results for this spectral sequence as well as its dual homotopy colimit spectral sequence. As applications, we obtain Artin and Brauer type induction theorems for G-equivariant E-homology and cohomology, and generalizations of Quillen's $\mathcal{F}_p$-isomorphism theorem when E is a homotopy commutative G-ring spectrum.

We show that the subcategory $\mathscr{F}^{\mathrm{Nil}}$ contains many G-spectra of interest for relatively small families $\mathscr{F}$. These include G-equivariant real and complex K-theory as well as the Borel-equivariant cohomology theories associated to complex oriented ring spectra, the L_n-local sphere, the classical bordism theories, connective real K-theory, and any of the standard variants of topological modular forms. In each of these cases we identify the minimal family such that these results hold.

Contents

Date: Tuesday 26^{th} April, 2016.

The first author was supported by the NSF Graduate Research Fellowship under grant DGE-110640. The second author was partially supported by the SFB 1085 - Higher Invariants, Regensburg. The third author was partially supported by the DFG grants: NO 1175/1-1 and SFB 1085 - Higher Invariants, Regsensburg.

1. Introduction

1.1. **Motivation and overview.** Let G be a finite group and $R(G)$ the Grothendieck ring of finite-dimensional complex representations of G. One can ask if $R(G)$ is determined by the representation rings $R(H)$ as H varies over some set $\mathcal{C}$ of subgroups of G. For example, every G-representation V has an underlying, or restricted, H-representation $\mathrm{Res}_H^G V$, and we can ask if the product of the restriction maps

$$\mathrm{Res}_{\mathcal{C}}^G : R(G) \longrightarrow \prod_{H \in \mathcal{C}} R(H)$$

is injective. By elementary character theory, this holds if $\mathcal{C}$ contains the cyclic subgroups of G.

Alternatively, associated to every H-representation W is an induced G-representation $\mathrm{Ind}_H^G W$ and we can ask if the direct sum of the induction maps

$$\mathrm{Ind}_{\mathcal{C}}^G : \bigoplus_{H \in \mathcal{C}} R(H) \longrightarrow R(G)$$

is surjective. This holds if $\mathcal{C}$ contains the Brauer elementary subgroups of G, i.e., subgroups of the form $C \times P$, where P is a p-group and C is a cyclic group of order prime to p [Ser77, §10.5, Thm. 19].

In general, there are strong restrictions on elements in the image of the restriction homomorphism: for example, an element $\{W_H\} \in \prod_{H \in \mathcal{C}} R(H)$ can only be in the image of $\mathrm{Res}_{\mathcal{C}}^G$ if

(1) for every pair of subgroups $H_1, H_2 \in \mathcal{C}$ such that $H_1 \leq H_2$, $\mathrm{Res}_{H_1}^{H_2} W_{H_2} = W_{H_1}$ and
(2) for every pair of subgroups $H_1, H_2 \in \mathcal{C}$ and $g \in G$ such that $gH_1g^{-1} = H_2$, W_{H_2} is the image of W_{H_1} under the isomorphism $R(H_1) \xrightarrow{\sim} R(H_2)$ induced by conjugating H_1 by g.

As a result, one expands $\mathcal{C}$ to a *family* $\mathscr{F}$ of subgroups, that is, a nonempty collection of subgroups closed under subconjugation. Then one can consider the subset of the product $\prod_{H \in \mathscr{F}} R(H)$ consisting of those elements which satisfy conditions (1) and (2). This subset can be identified with a certain limit, $\lim_{\mathcal{O}(G)_{\mathcal{F}}^{\mathrm{op}}} R(H)$, indexed over a subcategory $\mathcal{O}(G)_{\mathcal{F}}$ of the orbit category of G, and the restriction map naturally lifts to this limit.

We can apply a dual construction for the induction homomorphism to obtain maps which factor through the induction and restriction maps above:

$$\bigoplus_{G/H\in\mathcal{O}(G)_{\mathcal{F}}} R(H) \twoheadrightarrow \operatorname*{colim}_{\mathcal{O}(G)_{\mathcal{F}}} R(H) \xrightarrow{\mathrm{Ind}_{\mathcal{F}}^{G}} R(G) \xrightarrow{\mathrm{Res}_{\mathcal{F}}^{G}} \lim_{\mathcal{O}(G)_{\mathcal{F}}^{\mathrm{op}}} R(H) \hookrightarrow \prod_{G/H\in\mathcal{O}(G)_{\mathcal{F}}^{\mathrm{op}}} R(H). \tag{1.1}$$

If $\mathcal{F}$ is a family of subgroups which contains the Brauer elementary subgroups, then both $\mathrm{Ind}_{\mathcal{F}}^{G}$ and $\mathrm{Res}_{\mathcal{F}}^{G}$ are isomorphisms.[1] If instead we set $\mathcal{F}$ to be the generally smaller family of cyclic subgroups, these maps are isomorphisms after inverting the order of G. We can regard these two results as forms of the induction/restriction theorems of Brauer and Artin respectively [Ser77, Chaps. 9-10].

The discussion above formally extends to the study of Mackey functors of G. A *Mackey functor* M assigns an abelian group $M(H)$ to each subgroup $H \leq G$. These abelian groups are related by induction, restriction, and conjugation maps satisfying certain identities. In the theory of Mackey functors, one aims to find the smallest family $\mathcal{F}$ of subgroups of G for a given M such that we can reconstruct $M(G)$ from $M(H)$ as H varies over $\mathcal{F}$ as in Brauer's theorem [Dre73]. Such a family is called the *defect base* of M.

Recall that Mackey functors naturally occur as the homotopy groups of G-spectra. For example, $R(G)$ is the zeroth homotopy group of the G-fixed point spectrum of equivariant K-theory, $R(G) \cong \pi_0^G KU$. Given a G-spectrum M and a subgroup $H \leq G$, we associate the G-spectra

$$G/H_+ \wedge M \simeq F(G/H_+, M);$$

we have $\pi_0^G(G/H_+ \wedge M) \cong \pi_0^H M \cong \pi_0^G F(G/H_+, M)$. As G/H varies over the orbit category of G, the covariant (resp. contravariant) functoriality of $G/H_+ \wedge M$ (resp. $F(G/H_+, M)$) gives the induction (resp. restriction) maps in the Mackey functor $\pi_0^{(-)} M$.

By taking *homotopy* colimits and limits instead, we can obtain *derived* analogues of the maps in (1.1) for a G-spectrum M:

$$\operatorname*{hocolim}_{\mathcal{O}(G)_{\mathcal{F}}} G/H_+ \wedge M \xrightarrow{\mathrm{Ind}_{\mathcal{F}}^{G}} M \xrightarrow{\mathrm{Res}_{\mathcal{F}}^{G}} \operatorname*{holim}_{\mathcal{O}(G)_{\mathcal{F}}^{\mathrm{op}}} F(G/H_+, M). \tag{1.2}$$

We can now ask when $\mathrm{Ind}_{\mathcal{F}}^{G}$ and $\mathrm{Res}_{\mathcal{F}}^{G}$ are equivalences. Below we will study a stronger condition on M, namely that it should be *$\mathcal{F}$-nilpotent.* This will insure that not only are these maps equivalences, but also that the corresponding homotopy colimit and limit spectral sequences collapse in a particularly strong way. On homotopy groups, this will imply an analogue of Artin's theorem (see Theorem B).

Now if $M = R$ is a homotopy commutative G-ring spectrum, then the restriction maps are maps of ring spectra such that the lift $\mathrm{Res}_{\mathcal{F}}^{G}$ is a ring homomorphism, and we get a corresponding map of graded commutative rings after applying π_*^G. For example, if $R = \underline{H\mathbb{F}_p}$ is the G-spectrum representing mod-p Borel-equivariant cohomology, then we obtain a ring homomorphism

$$\pi_{-*}^G \underline{H\mathbb{F}_p} \cong H^*(BG; \mathbb{F}_p) \xrightarrow{\mathrm{Res}_{\mathcal{F}}^{G}} \lim_{\mathcal{O}(G)_{\mathcal{F}}^{\mathrm{op}}} H^*(BH; \mathbb{F}_p).$$

A celebrated result of Quillen [Qui71, Thm. 7.1] states that this map is a uniform $\mathcal{F}_p$-isomorphism when $\mathcal{F} = \mathcal{E}_{(p)}$ is the family of elementary abelian p-subgroups of G, i.e., subgroups of the form $C_p^{\times n}$ for some non-negative integer n. Recall that a ring map $f\colon A \to B$ is a uniform $\mathcal{F}_p$-isomorphism if there are integers $m > 0$ and $n \geq 0$ such that if $x \in \ker f$ and $y \in B$ then $x^m = 0$

[1]In fact, these maps are isomorphisms if and only if $\mathcal{F}$ contains the Brauer elementary subgroups [Ser77, §11.3, Thm. 23"'].

and $y^{p^n} \in \mathrm{Im} f$. We will see that $\underline{H\mathbb{F}_p}$ is $\mathscr{E}_{(p)}$-nilpotent and that our collapse results for the homotopy limit spectral sequence imply Quillen's theorem as well as analogs for every homotopy commutative $\mathscr{F}$-nilpotent G-ring spectrum (see Theorem C).

1.2. **Main results.** Throughout this paper, G will denote a finite group and $\mathscr{F}$ a family of subgroups of G. We will work in the stable presentable ∞-category of G-spectra Sp_G equipped with its symmetric monoidal smash product [MNN, Part 2]. However, if the reader prefers, they can recast our work in the setting of model categories and use [MM02] or [Man04]. Of course, one will still need a theory of ∞-categories, as developed in [Lur09, Lur14], for descent applications such as [MNN, Thm. 6.42].

The focus of this paper is the following subcategory of G-spectra.

Definition 1.3 (Cf. [MNN, Def. 6.36])**.** Let $\mathscr{F}^{\mathrm{Nil}}$, the ∞-category of *$\mathscr{F}$-nilpotent* G-spectra, be the smallest thick $\otimes$-ideal in Sp_G containing $\{G/H_+\}_{H\in\mathscr{F}}$. In other words, $\mathscr{F}^{\mathrm{Nil}}$ is the smallest full subcategory of Sp_G such that:

(1) For each subgroup $H \in \mathscr{F}$, the suspension G-spectrum G/H_+ is $\mathscr{F}$-nilpotent.
(2) For $E, F \in \mathrm{Sp}_G$ and $f \in \mathrm{Sp}_G(E, F)$, let Cf denote the cofiber of f. If any two of $\{E, F, Cf\}$ are $\mathscr{F}$-nilpotent, then all three of them are $\mathscr{F}$-nilpotent.
(3) If $E \in \mathrm{Sp}_G$ is a retract of an $\mathscr{F}$-nilpotent G-spectrum, then E is $\mathscr{F}$-nilpotent.
(4) If $E \in \mathrm{Sp}_G$ and F is $\mathscr{F}$-nilpotent, then $E \wedge F$ is $\mathscr{F}$-nilpotent.

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two families of subgroups of G and let $\mathscr{F}_1 \cap \mathscr{F}_2$ denote their intersection. Then $\mathscr{F}_1^{\mathrm{Nil}} \cap \mathscr{F}_2^{\mathrm{Nil}} = (\mathscr{F}_1 \cap \mathscr{F}_2)^{\mathrm{Nil}}$ by [MNN, Prop. 6.39]. For any G-spectrum M, there is thus a minimal family $\mathscr{F}$ such that M is $\mathscr{F}$-nilpotent; we will call this minimal family the *derived defect base* of M.

Although the above definition is simple, it is generally difficult to determine the derived defect base directly. We will provide several alternative characterizations of $\mathscr{F}^{\mathrm{Nil}}$ shortly. First we recall some notation.

For a real orthogonal representation V of G, let S^V denote the one-point compactification, considered as a pointed G-space with ∞ as basepoint. The inclusion $0 \subset V$ induces an equivariant map $e_V\colon S^0 \to S^V$ called the *Euler class* of V. We consider in particular the case $V = \tilde{\rho}_G$, the reduced regular representation of G.

Theorem A. Let $M \in \mathrm{Sp}_G$. The following three conditions on M are equivalent:

(1) The G-spectrum M is $\mathscr{F}$-nilpotent.
(2) For each subgroup $K \notin \mathscr{F}$, $e_{\tilde{\rho}_K}$ is a nilpotent endomorphism of the K-spectrum $\mathrm{Res}^G_K M$. In other words, there exists $n \geq 0$ such that the map $e_{n\tilde{\rho}_K} \simeq e^n_{\tilde{\rho}_K}$ is null-homotopic after smashing with $\mathrm{Res}^G_K M$.
(3) The map of G-spectra $\mathrm{Res}^G_{\mathscr{F}} : M \to \mathrm{holim}_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} F(G/H_+, M)$ of (1.2) is an equivalence and there is an integer $n \geq 0$ such that for every G-spectrum X, the $\mathscr{F}$-homotopy limit spectral sequence:
$$E_2^{s,t} = \lim^s_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi^H_t F(X, M) \Longrightarrow \pi^G_{t-s} F(X, \operatorname*{holim}_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} F(G/H_+, M)) \cong M^{s-t}_G(X)$$
has a horizontal vanishing line of height n on the E_{n+1}-page. In other words, we have $E^{k,*}_{n+1} = 0$ for all $k > n$.

Theorem A is fundamental to this paper. Condition (2) is often easy to check in practice, especially in the presence of *Thom isomorphisms* for representation spheres (see Section 5 for some examples); these will lead to most of our examples of $\mathscr{F}$-nilpotence.

The equivalence between Conditions (1) and (3) has several computational consequences which we will now list. The first two results follow from the horizontal vanishing line and a general result of Amitsur-Dress-Tate theory which implies that the elements in positive filtration degree in the hocolim and holim spectral sequences are $|G|$-torsion.

Theorem B. Let M and X be G-spectra. Suppose that M is $\mathscr{F}$-nilpotent. Then each of the following maps

$$\operatorname*{colim}_{\mathcal{O}(G)_{\mathscr{F}}} M_H^*(X) \xrightarrow{\operatorname{Ind}_{\mathscr{F}}^G} M_G^*(X) \xrightarrow{\operatorname{Res}_{\mathscr{F}}^G} \lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M_H^*(X)$$

$$\operatorname*{colim}_{\mathcal{O}(G)_{\mathscr{F}}} M_*^H(X) \xrightarrow{\operatorname{Ind}_{\mathscr{F}}^G} M_*^G(X) \xrightarrow{\operatorname{Res}_{\mathscr{F}}^G} \lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M_*^H(X)$$

becomes an isomorphism after inverting $|G|$.

We next state our general analog of Quillen's $\mathcal{F}_p$-isomorphism theorem.

Theorem C. Let R be a homotopy commutative G-ring spectrum and let X be a G-space. Suppose that R is $\mathscr{F}$-nilpotent. Then the canonical map

$$R_G^*(X) \xrightarrow{\operatorname{Res}_{\mathscr{F}}^G} \lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^*(X)$$

is a uniform $\mathcal{N}$-isomorphism[2]: there are positive integers m, n such that if $x \in \ker \operatorname{Res}_{\mathscr{F}}^G$ and $y \in \lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^*(X)$ then $x^m = 0$ and $y^n \in \operatorname{Im} \operatorname{Res}_{\mathscr{F}}^G$. Moreover, after localizing at a prime p, $\operatorname{Res}_{\mathscr{F}}^G$ is a uniform $\mathcal{F}_p$-isomorphism.

Corollary 1.4. Under the hypotheses of Theorem C, the map $\operatorname{Res}_{\mathscr{F}}^G : R_G^0(X) \to \lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^0(X)$ induces a homeomorphism between the associated Zariski spaces[3]:

$$\operatorname{Spec}\left(\lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^0(X) \right) \to \operatorname{Spec}(R_G^0(X)).$$

For $M \in \mathscr{F}^{\mathrm{Nil}}$, the minimal integer n satisfying Theorem A, Item 3 is called the $\mathscr{F}$-exponent of M. We include various characterizations of this numerical invariant below.

We can identify the derived defect bases for many G-equivariant ring spectra of interest. These are listed in Figure 1.5B, where we set the notation for the relevant families of subgroups in Figure 1.5A. Many of these examples arise from non-equivariant ring spectra by taking their associated Borel theories as in [MNN, §6.3]. There, as above, we are letting $\underline{M}$ denote the *Borel-equivariant G-spectrum associated to a spectrum M* with a G-action. All of the examples above come from spectra with trivial G-actions, in which case this equivariant cohomology theory theory is defined so that, for a G-spectrum X,

$$\underline{M}_G^*(X) = M^*(EG_+ \wedge_G X).$$

We now draw the reader's attention to a few patterns appearing in Figure 1.5B. First, if $R \in \mathrm{Sp}_G$ is $\mathscr{F}$-nilpotent, then its Borel completion $\underline{R}$ is automatically $\underline{\mathscr{F}}$-nilpotent (i.e., we only need to consider the p-groups in $\mathscr{F}$, as p varies over the primes dividing $|G|$). The notation respects localization in the following sense: if $\underline{R}$ is $\underline{\mathscr{F}}$-nilpotent, then $\underline{R_{(p)}}$ (resp. $\underline{R[1/n]}$) will automatically be $\mathscr{F}_{(p)} = \underline{\mathscr{F}}_{(p)}$-nilpotent (resp. $\underline{\mathscr{F}}[1/n]$-nilpotent). These results are immediate consequences of Theorem 4.25 and allow one to determine derived defect bases for Borel-equivariant G-spectra via arithmetic fracture square arguments.

Perhaps the most prominent pattern occurring in Figure 1.5B is the connection between the 'chromatic complexity' of a G-spectrum E and the complexity of E's derived defect base. The

[2]We believe this term was first coined in [Hop87, p. 88].

[3]Under additional finiteness hypotheses (see Proposition 3.27), there is a further identification: $\operatorname{colim}_{\mathcal{O}(G)_{\mathscr{F}}} \operatorname{Spec}(R_H^0(X)) \cong \operatorname{Spec}(\lim_{\mathcal{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^0(X))$.

Notation	Definition of family
$\mathscr{A}\ell\ell$	All subgroups
$\mathscr{P}$	Proper subgroups
$\mathscr{T}$	Only the trivial subgroup
$\mathscr{A}$	Abelian subgroups
$\mathscr{A}^n$	Abelian subgroups which can be generated by n elements
$\mathscr{C} = \mathscr{A}^1$	Cyclic subgroups
$\mathscr{E}$	Subgroups of the form $C_p^{\times n}$ for some prime p and some n
$\mathscr{F}(K)$	Subgroups in $\mathscr{F}$ which are subconjugate to $K \leq G$
$\underline{\mathscr{F}}$	Subgroups H in $\mathscr{F}$ such that $\lvert H\rvert = p^n$ for some prime p and some n
$\mathscr{F}_{(p)}$	Subgroups H in $\mathscr{F}$ such that $\lvert H\rvert = p^n$ for some n
$\mathscr{F}[1/n]$	Subgroups H in $\mathscr{F}$ such that $n \nmid \lvert H\rvert$
$\mathscr{F}_1 \cup \mathscr{F}_2$	Subgroups H in either $\mathscr{F}_1$ or $\mathscr{F}_2$

(A) Families of subgroups.

G-Spectrum R	Derived defect base	Proof of claim
$S, S\otimes\mathbb{Q}$	$\mathscr{A}\ell\ell$	Proposition 4.22
$K\mathbb{R}$ $(G = C_2)$	$\mathscr{T}$	Proposition 2.14
$M\mathbb{R}$ $(G = C_2)$	$\mathscr{A}\ell\ell$	Proposition 2.15
MU, MO	$\mathscr{A}\ell\ell$	Proposition 2.15
$H\mathbb{Z}$	$\underline{\mathscr{A}\ell\ell}$	Proposition 4.24
$H\mathbb{Q}$	$\mathscr{T}$	Proposition 4.24
KO, KU	$\mathscr{C}$	Proposition 5.6
ko, ku	$\mathscr{C}\cup\mathscr{E}$	Proposition 5.11
$\underline{S}$	$\underline{\mathscr{A}\ell\ell}$	Theorem 4.25
$\underline{S\otimes\mathbb{Q}}$	$\mathscr{T}$	Theorem 4.25
$\underline{MU}$	$\underline{\mathscr{A}}$	Theorem 5.14
$\underline{H\mathbb{F}_p}$	$\mathscr{E}_{(p)}$	Proposition 5.16
$\underline{H\mathbb{Z}}$	$\mathscr{E}$	Proposition 5.25
$\underline{ku}$	$\mathscr{E}\cup\underline{\mathscr{C}}$	Corollary 5.34
$\underline{BP\langle n\rangle}$	$\mathscr{E}_{(p)}\cup\mathscr{A}^n_{(p)}$	Proposition 5.32
$\underline{K(n)}$	$\mathscr{T}$	Proposition 5.31
$\underline{T(n)}$	$\mathscr{T}$	Proposition 5.31
$\underline{E_n}$	$\mathscr{A}^n_{(p)}$	Proposition 5.26
$\underline{L_n S}$	$\mathscr{A}^n_{(p)}$	Proposition 5.36
$\underline{ko}$	$\mathscr{E}\cup\underline{\mathscr{C}}$	Proposition 5.38
$\underline{KO}, \underline{KU}$	$\underline{\mathscr{C}}$	Proposition 5.38
$\underline{Tmf}, \underline{TMF}$	$\underline{\mathscr{A}}^2$	Proposition 5.40
$\underline{tmf}$	$\mathscr{E}\cup\underline{\mathscr{A}}^2$	Proposition 5.41
$\underline{MO}$	$\mathscr{E}_{(2)}$	Corollary 5.17
$\underline{MSO}$	$\mathscr{E}_{(2)}\cup\underline{\mathscr{A}}[1/2]$	Proposition 5.42
$\underline{MSp[1/2]}$	$\underline{\mathscr{A}}[1/2]$	Corollary 5.44
$\underline{MSpin}$	$\mathscr{E}_{(2)}\cup\mathscr{C}_{(2)}\cup\underline{\mathscr{A}}[1/2]$	Proposition 5.45
$\underline{MString[1/2]}$	$\underline{\mathscr{A}}[1/2]$	Proposition 5.46
$\underline{MU\langle 6\rangle}$	$\underline{\mathscr{A}}$	Proposition 5.47

(B) Derived defect bases for some G-ring spectra.

FIGURE 1.5

informal slogan is 'if E is sensitive to height n chromatic phenomena, then E should be nilpotent for the family of abelian subgroups which can be generated by n or fewer elements'. This slogan is motivated by the generalized character theory of [HKR00]. That theory and the Hopkins-Ravenel smash product theorem let us make this slogan precise in Proposition 5.36: if E is L_n-local, then $\underline{E}$ is $\mathscr{A}^n_{(p)}$-nilpotent.

1.3. **Related work.** Special cases of Theorem B and Theorem C have been proven under additional hypotheses by various authors:

- Segal proves the analog of Theorem C for G-equivariant K-theory for a general compact Lie group when X is a point and $\mathscr{F}$ is the family of topologically cyclic subgroups with finite Weyl groups [Seg68, Prop. 3.5]. Segal also proves an analog of Brauer's theorem in this setting [Seg68, Prop. 3.11].
- The most celebrated form of Theorem C is [Qui71, Thm. 6.2]. There, Quillen proves this result in the case $M = \underline{H\mathbb{F}_p}$, $\mathscr{F} = \mathscr{E}_{(p)}$, G is a compact Lie group, and X is a G-CW complex of finite mod-p cohomological dimension. In the case $X = *$, Quillen also proves

this result in the case G is a compact Hausdorff group with only finitely many conjugacy classes of elementary abelian subgroups [Qui71, Prop. 13.4], along with an extension to the case where G is a discrete subgroup with a finite index subgroup H of finite mod-p cohomological dimension [Qui71, Thm. 14.1].

Quillen's seminal work underlies all of the following research in this direction including our own. This paper owes a tremendous debt to him.

- Bojanowska and Jackowski prove Theorem C in the case $M = KU$, $\mathscr{F} = \mathscr{C}$, and X is a finite G-CW complex. They also prove that the homotopy limit spectral sequence has the desired abutment [BJ80].
- Greenlees and Strickland prove a result similar to Corollary 1.4 in the case that $M = \underline{E}$, E is a complex oriented ring spectrum with formal properties similar to E_n, X is a finite G-CW complex, and $\mathscr{F} = \mathscr{A}^n_{(p)}$ [GS99, Thm. 3.5]. They also obtain suitable extensions when G is a compact Lie group [GS99, App. C].
- Hopkins, Kuhn, and Ravenel prove Theorem B in the case where $M = \underline{E}$, E is a complex oriented ring spectrum, $\mathscr{F} = \mathscr{A}$, and X is a finite G-CW complex [HKR00, Thm. A and Rem. 3.5].
- In [Fau08] Fausk shows that [HKR00, Thm. A] can be generalized in several ways if one makes some additional assumptions. First, Fausk proves the analogue of Theorem B when $M = KU$ and G is a compact Lie group. Moreover, Fausk proves Theorem B when $M = \underline{E_n}$ (or a closely related ring spectrum), G is a finite group, $\mathscr{F} = \mathscr{A}^n_{(p)}$, and $\pi^G_* M$ is torsion-free (e.g., when G is a good group in the sense of [HKR00]). Fausk also obtains generalized Brauer induction theorems in these contexts. Fausk's results do not require a finiteness assumption on X.

We note that with the exception of [Fau08], all of the above results require some finiteness hypothesis on X. Our results do not require this condition, but this comes at a cost: under additional finiteness assumptions the authors above often obtain a simpler description of $M^*_G(X)$ up to $\mathcal{F}_p$-isomorphism. As a functor of the G-space X, these simpler descriptions only depend on $\pi_0 X^{(-)}$.

In light of the above classical results, it is natural to ask if our methods can be applied to the study of G-spectra when G is a general compact Lie group. Our results, which depend on the analysis of the homotopy limit spectral sequence, immediately break down in this case, but many of the other statements go through with only obvious changes in the case of compact Lie groups. We hope to provide a detailed discussion at a later time.

The homotopy limit spectral sequence in Theorem A can be constructed as a descent spectral sequence associated to the map of E_∞-algebras $S \to \prod_{H \in \mathscr{F}} F(G/H_+, S)$ constructed from the restriction maps. After smashing with an $\mathscr{F}$-nilpotent E_∞-algebra in Sp_G, one obtains a map admitting 'descent up to nilpotence' in the sense of [Bal16]. That paper and its treatment of Quillen's stratification theorem [Qui71, Thm. 10.2] were a significant inspiration for this work.

Organization. In Section 2, we will analyze the class $\mathscr{F}^{\mathrm{Nil}}$ of G-spectra and prove Theorem A. We break this proof into two parts. In Section 2.1, we prove the equivalence of Conditions (1) and (2) of Theorem A (Theorem 2.3) as well as some immediate consequences. In Section 2.3, we prove the equivalence of Conditions (1) and (3) (Theorem 2.25).

In Section 3, we will analyze the homotopy colimit and homotopy limit spectral sequences. This will lead to proofs of Theorem B, Theorem C, and Corollary 1.4 in Section 3.3. Along the way we will prove Proposition 3.27, which is the appropriate analogue of Quillen's stratification theorem [Qui71, Thm. 8.10] in this context.

In Section 4 we show that derived induction and restriction theory generalizes classical induction and restriction theory and reduces to it exactly for $\mathscr{F}$-nilpotent spectra of exponent at most one. We show that one can use the calculation of the derived defect base of a G-ring spectrum to put an upper bound on its defect base (Proposition 4.12). As applications, we obtain a generalized hyperelementary induction theorem similar to Brauer's theorem (Theorem 4.16) and triangulated descent results in the sense of Balmer (Proposition 4.19).

In the last two sections of the main body, we prove all of the remaining claims in Figure 1.5B. In Section 5, we show how the existence of Thom isomorphisms can be used to show a G-ring spectrum is $\mathscr{F}$-nilpotent. We then combine these results with non-equivariant thick subcategory arguments to determine the derived defect bases of the remaining examples.

In the appendices we gather several auxiliary results for working with the $\mathscr{F}$-homotopy limit spectral sequences and work through a nontrivial example for equivariant topological K-theory.

Acknowledgments. The authors would like to thank John Greenlees, Hans-Werner Henn, Mike Hopkins, Peter May, Charles Rezk, and Nat Stapleton for helpful comments related to this project. We would also like to thank Koen van Woerden for proofreading an earlier draft of this paper. Finally we would like to thank the referee for providing helpful remarks on this paper.

Conventions. Throughout this paper, G will denote a finite group and $\mathscr{F}$ a family of subgroups of G. For two G-spectra X and Y we will let $F(X,Y) \in \mathrm{Sp}_G$ denote the internal function G-spectrum. Unless we believe it to be helpful to the reader, we will generally suppress the functors Σ^∞ and Res^G_K from our notation.

A G-ring spectrum R will always be a G-spectrum equipped with a homotopy associative and unital multiplication, i.e., an associative algebra in $\mathrm{Ho}(\mathrm{Sp}_G)$. We will say that R is homotopy commutative if the multiplication is commutative in $\mathrm{Ho}(\mathrm{Sp}_G)$. An R-module will be an object of $\mathrm{Ho}(\mathrm{Sp}_G)$ equipped with a left action of R satisfying the standard associativity and unit conditions. We will use the adjective *structured* when we want to talk about the ∞-categorical or model categorical notion of module.

2. The thick $\otimes$-ideal $\mathscr{F}^{\mathrm{Nil}}$

In this section, we give the main characterizations of $\mathscr{F}$-nilpotence and prove Theorem A.

2.1. The characterization of $\mathscr{F}^{\mathbf{Nil}}$ in terms of Euler classes.

In this subsection, we will prove the equivalence of Conditions (1) and (2) from Theorem A in Theorem 2.3 below. First, we will require some elementary properties of representation spheres.

Definition 2.1. For a finite-dimensional orthogonal representation V of G, we let nV denote $V^{\oplus n}$. We let $S(V)$ denote the unit G-sphere of V and S^V denote the pointed G-space obtained as the one-point compactification of V, where we take the point at ∞ to be the basepoint. Finally, we let e_V, the *Euler class* of V, denote the pointed G-map

$$e_V \colon S^0 \to S^V$$

induced by the inclusion $0 \to V$.

We now recall the following standard results.

Proposition 2.2. Let V be a finite-dimensional orthogonal representation of G. Then:

(1) If V contains a trivial summand, then e_V is G-equivariantly homotopic to the trivial map.
(2) The G-space S^V is the cofiber of the nontrivial map $S(V)_+ \to S^0$.

(3) The G-space $S(V)$ admits a finite G-CW structure constructed from cells of the form

$$G/H \times S^n \to G/H \times D^{n+1}$$

where H is a subgroup such that $V^H \neq \{0\}$ and $n < \dim V^H$. Compare [tD87, Exer. II.1.. II.1.10] and [Ill83].

(4) For every $n \geq 0$, we have $e_V^n \simeq e_{nV}$.

We now prove the main characterization of $\mathscr{F}$-nilpotence in terms of Euler classes.

Theorem 2.3. A G-spectrum M is $\mathscr{F}$-nilpotent if and only if, for all subgroups $K \leq G$ with $K \notin \mathscr{F}$, there exists an integer n such that the Euler class $e_{n\tilde{\rho}_K} : S^0 \to S^{n\tilde{\rho}_K}$ is null-homotopic after smashing with $\mathrm{Res}^G_K M$.

Proof. Let $\mathscr{F}^{\mathrm{Nil}'} \subseteq \mathrm{Sp}_G$ denote the full subcategory spanned by the $M \in \mathrm{Sp}_G$ satisfying the Euler class condition of the theorem. It is easy to see that $\mathscr{F}^{\mathrm{Nil}'}$ is a thick $\otimes$-ideal. We need to show that $\mathscr{F}^{\mathrm{Nil}} = \mathscr{F}^{\mathrm{Nil}'}$.

For a subgroup $H \leq G$, let $\mathcal{P}_H$ denote the family of proper subgroups of H. Observe that $M \in \mathscr{F}^{\mathrm{Nil}'}$ if and only if, for every $H \leq G$ not in $\mathscr{F}$, we have $\mathrm{Res}^G_H M \in \mathcal{P}_H^{\mathrm{Nil}'}$. Moreover, one has a similar statement for $\mathscr{F}$-nilpotence: by [MNN, Prop. 6.40], $M \in \mathscr{F}^{\mathrm{Nil}}$ if and only if for every subgroup $H \notin \mathscr{F}$, $\mathrm{Res}^G_H M \in \mathcal{P}_H^{\mathrm{Nil}}$.

It thus suffices to consider the case where $\mathscr{F} = \mathcal{P}_G$. In other words, we need to show that the thick $\otimes$-ideal generated by $\{G/H_+\}_{H<G}$ is equal to $\mathcal{P}_G^{\mathrm{Nil}'}$. Observe first that the Euler class $e_{\widetilde{\rho}_G}$ becomes null-homotopic after smashing with G/H_+ for any $H < G$. This follows because for any $H < G$, $\mathrm{Res}^G_H e_{\widetilde{\rho}_G}$ is null-homotopic as the H-representation $\mathrm{Res}^G_H \widetilde{\rho}_G$ contains a trivial summand. Here we use the relationship between smashing with G/H_+ and restricting to H-spectra; compare [BDS15, Thm. 1.1] and [MNN, Thm. 5.32]. Therefore, we get $G/H_+ \in \mathcal{P}_G^{\mathrm{Nil}'}$, so that $\mathcal{P}_G^{\mathrm{Nil}} \subset \mathcal{P}_G^{\mathrm{Nil}'}$.

We now prove the opposite inclusion. Suppose $M \in \mathcal{P}_G^{\mathrm{Nil}'}$. Then there exists n such that $\mathrm{id}_M \wedge e_{n\widetilde{\rho}_G}$ is null-homotopic, and the cofiber sequence

$$S(n\widetilde{\rho}_G)_+ \wedge M \to M \xrightarrow{\mathrm{id}_M \wedge e_{n\widetilde{\rho}_G}} M \wedge S^{n\widetilde{\rho}_G}$$

shows that M is a retract of $S(n\widetilde{\rho}_G)_+ \wedge M$. Since $\widetilde{\rho}_G$ has no non-trivial fixed points, $S(n\widetilde{\rho}_G)_+ \in \mathcal{P}_G^{\mathrm{Nil}}$ in view of the cell decomposition given in Proposition 2.2. Therefore, $S(n\widetilde{\rho}_G)_+ \wedge M$ is $\mathcal{P}_G$-nilpotent, and thus its retract M is too. □

Remark 2.4. If we regard $e_{\tilde{\rho}_K}$ as an element of the '$RO(K)$-graded homotopy groups' [Ada84, §6], $\pi^K_\star S$, of the sphere spectrum, then after smashing $e_{\tilde{\rho}_K}$ with M we obtain an element in $\pi^K_\star F(M,M)$, the '$RO(K)$-graded homotopy groups' of the endomorphism ring of M. This element can also be identified with the image of $e_{\tilde{\rho}_K}$ under the unit map $S \to F(M,M)$.

Identifying $e_{\tilde{\rho}_K}$ with its image, we can now restate the null-homotopy condition of Theorem 2.3 in either of the following equivalent ways:

(1) $e_{\tilde{\rho}_K} \in \pi^K_\star F(M,M)$ is nilpotent, or
(2) $F(M,M)[e^{-1}_{\tilde{\rho}_K}] \simeq * \in \mathrm{Sp}_K$.

While $M \in \mathscr{F}^{\mathrm{Nil}}$ implies that $M[e^{-1}_{\tilde{\rho}_K}]$ is contractible for each $K \notin \mathscr{F}$, the converse does not hold. The contractibility of $M[e^{-1}_{\tilde{\rho}_K}]$ is equivalent to knowing that every element $x \in \pi^K_\star M$ is annihilated by *some* power, possibly depending on x, of $e_{\tilde{\rho}_K}$. The condition $M \in \mathscr{F}^{\mathrm{Nil}}$ tells us that there is a *fixed* power of $e_{\tilde{\rho}_K}$ which annihilates all of $\pi^K_\star M$.

On the other hand, when $M = R$ is a G-ring spectrum, the two conditions are equivalent because the power of $e_{\tilde{\rho}_K}$ annihilating $1 \in \pi^K_* R$ annihilates all of $\pi^K_* R$.

Corollary 2.5. Suppose that R is a G-ring spectrum. Then the following are equivalent:

(1) The G-spectrum R is $\mathscr{F}$-nilpotent.
(2) For each subgroup $H \notin \mathscr{F}$, the image of $e_{\tilde{\rho}_H} \in \pi_\star^H S$ under the unit map $S \to R$ is nilpotent.

2.2. **The $\mathscr{F}$-homotopy limits and colimits.** We will now precisely define the homotopy colimits and limits mentioned in the introduction in (1.2) and prove they are equivalences when M is $\mathscr{F}$-nilpotent.

We denote the category of G-spaces by $\mathcal{S}_G$. As usual, let $\mathscr{O}(G) \subseteq \mathcal{S}_G$ (the *orbit category*) denote the full subcategory of $\mathcal{S}_G$ spanned by the transitive G-sets. To a family $\mathscr{F}$ we associate the full subcategory $\mathscr{O}(G)_{\mathscr{F}} \subset \mathscr{O}(G)$ spanned by the transitive G-sets whose isotropy lies in $\mathscr{F}$.

Let $i\colon \mathscr{O}(G)_{\mathscr{F}} \to \mathcal{S}_G$ denote the inclusion. We associate to $\mathscr{F}$ a G-space $E\mathscr{F} := \operatorname{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} i$. We also define a pointed G-space $\widetilde{E\mathscr{F}}$ as the homotopy cofiber of the unique nontrivial map $E\mathscr{F}_+ \to S^0$. These G-spaces are determined up to canonical equivalence by the following properties (see Appendix A.1 and [LMS86, Def. II.2.10]):

$$E\mathscr{F}^K \simeq \begin{cases} * & \text{if } K \in \mathscr{F} \\ \emptyset & \text{otherwise} \end{cases} \qquad \widetilde{E\mathscr{F}}^K \simeq \begin{cases} * & \text{if } K \in \mathscr{F} \\ S^0 & \text{otherwise.} \end{cases} \tag{2.6}$$

For the family $\mathscr{P}$ of all proper subgroups of G, these spaces admit a particularly simple construction.

Proposition 2.7. There are canonical equivalences

$$E\mathscr{P} \simeq \operatorname*{hocolim}_n S(n\tilde{\rho}_G) \quad \text{and} \quad \widetilde{E\mathscr{P}} \simeq \operatorname*{hocolim}_n S^{n\tilde{\rho}_G} \simeq S[e_{\tilde{\rho}_G}^{-1}].$$

Here the homotopy colimits are indexed over the maps induced by the inclusions $n\tilde{\rho}_G \to (n+1)\tilde{\rho}_G$.

Proof. We just need to check that the homotopy colimits have the correct fixed points. Since fixed points commute with homotopy colimits, this follows from Proposition 2.2 and the following observation: $\tilde{\rho}_G^K$ is 0-dimensional if and only if $K = G$. □

We recall the significance of the objects $E\mathscr{F}_+, \widetilde{E\mathscr{F}}$ in the general theory, cf. [MNN, §6.1]. Let $\mathrm{Loc}_{\mathscr{F}}$ denote the *localizing* subcategory of Sp_G generated by the $\{G/H_+\}_{H\in\mathscr{F}}$. It is equivalently the subcategory of $A_{\mathscr{F}}$-torsion objects (cf. [MNN, Def 3.1]) with respect to the commutative algebra object

$$A_{\mathscr{F}} := \prod_{H\in\mathscr{F}} F(G/H_+, S) \in \mathrm{Sp}_G.$$

The inclusion $\mathrm{Loc}_{\mathscr{F}} \subset \mathrm{Sp}_G$ admits a right adjoint given by *$\mathscr{F}$-colocalization*; the right adjoint is given explicitly by $X \mapsto E\mathscr{F}_+ \wedge X$. In particular, $X \in \mathrm{Loc}_{\mathscr{F}}$ if and only if the natural map

$$E\mathscr{F}_+ \wedge X \to X$$

is an equivalence. We also have the subcategory of *$\mathscr{F}$-complete* G-spectra, i.e., those G-spectra complete with respect to the algebra object $A_{\mathscr{F}}$ [MNN, Sec. 2]. The G-space $E\mathscr{F}$ also controls the theory of $\mathscr{F}$-completeness: a G-spectrum X is $\mathscr{F}$-complete if and only if the natural map

$$X \to F(E\mathscr{F}_+, X)$$

is an equivalence.

We consider finally (cf. [MNN, Sec. 3.2]) the subcategory $\mathrm{Sp}_G[\mathscr{F}^{-1}]$ of those G-spectra Y such that $F(X, Y) \simeq *$ for any $X \in \mathrm{Loc}_{\mathscr{F}}$. Then $\mathrm{Sp}_G[\mathscr{F}^{-1}]$ is a *localization* of Sp_G, and the localization is given by the functor $X \mapsto \widetilde{E\mathscr{F}} \wedge X$. The localization functor annihilates *precisely* the localizing subcategory $\mathrm{Loc}_{\mathscr{F}}$. Note that, by definition [MNN, Def. 6.36], a G-spectrum is $\mathscr{F}$-nilpotent if and only if it is $A_{\mathscr{F}}$-nilpotent.

Using the general theory of torsion, complete, and nilpotent objects with respect to a *dualizable* algebra object, we now record the following list of properties of $\mathscr{F}^{\text{Nil}}$.

Proposition 2.8. (1) If M is an $\mathscr{F}$-nilpotent G-spectrum, then $\widetilde{E\mathscr{F}} \wedge M$ is contractible, and thus the map $M \wedge E\mathscr{F}_+ \to M$ is an equivalence. Similarly, the map $M \to F(E\mathscr{F}_+, M)$ is an equivalence.
(2) If M is a G-ring spectrum with $\widetilde{E\mathscr{F}} \wedge M$ contractible, then M is $\mathscr{F}$-nilpotent.
(3) Let X and M be G-spectra. If M is $\mathscr{F}$-nilpotent, then so is $F(X, M)$.
(4) A G-spectrum M is $\mathscr{F}$-nilpotent if and only if the endomorphism G-ring spectrum $F(M, M)$ is $\mathscr{F}$-nilpotent.

Proof. As above, a G-spectrum M belongs to the localizing subcategory $\text{Loc}_{\mathscr{F}}$ generated by the $\{G/H_+\}_{H\in\mathscr{F}}$ if and only if $M \wedge \widetilde{E\mathscr{F}}$ is contractible (or equivalently if $M \wedge E\mathscr{F}_+ \simeq M$). If M is $\mathscr{F}$-nilpotent, this is certainly the case. If $M \in \mathscr{F}^{\text{Nil}}$, then M is also complete with respect to the algebra object $A_{\mathscr{F}}$ so that the $\mathscr{F}$-completion map $M \to F(E\mathscr{F}_+, M)$ is an equivalence.

Conversely, if M is a G-ring spectrum, then the $\mathscr{F}^{-1}$-localization of M, i.e., $\widetilde{E\mathscr{F}} \wedge M$, vanishes if and only M is $\mathscr{F}$-nilpotent by [MNN, Thm. 4.18].

We refer to [MNN, Cor. 4.14] for the (general) argument that $\mathscr{F}^{\text{Nil}}$ is closed under cotensors. If $M \in \text{Sp}_G$ and $F(M, M) \in \mathscr{F}^{\text{Nil}}$, then M, as a module over $F(M, M)$, also belongs to $\mathscr{F}^{\text{Nil}}$. This verifies the third and fourth claims. □

We now construct the derived restriction and induction maps (1.2) in terms of the space $E\mathscr{F}$, as $\mathscr{F}$-colocalization and completion respectively.

Construction 2.9. We consider now the $\mathscr{F}$-colocalization map $E\mathscr{F}_+ \wedge M \to M$; since $E\mathscr{F} = \text{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} G/H_+$ and smash products commute with homotopy colimits, we can write this map as

$$\text{Ind}_{\mathscr{F}}^G \colon \operatorname*{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} (G/H_+ \wedge M) = E\mathscr{F}_+ \wedge M \to M. \tag{2.10}$$

Similarly, we can identify the $\mathscr{F}$-completion map $M \to F(E\mathscr{F}_+, M)$; with the map:

$$\text{Res}_{\mathscr{F}}^G \colon M \to \operatorname*{holim}_{\mathscr{O}(G)_{\mathscr{F}}^{op}} F(G/H_+, M). \tag{2.11}$$

Proposition 2.12. If M is $\mathscr{F}$-nilpotent, then the derived induction and restriction maps (2.10) and (2.11) are equivalences.

Proof. This now follows from Proposition 2.8. □

We round out this subsection with a few basic examples of derived defect bases. We remark also that this technique is essentially [HHR, Sec. 10].

Proposition 2.13. Let $\mathscr{F} = \mathscr{T}$ be the trivial family of subgroups. Suppose $\underline{R}$ is a Borel-equivariant G-ring spectrum. Then $\underline{R}$ is $\mathscr{T}$-nilpotent if and only if the G-Tate construction $(\widetilde{E\mathscr{T}} \wedge \underline{R})^G$ of R is contractible.

Proof. We know that $\underline{R}$ is $\mathscr{T}$-nilpotent if and only if $\widetilde{E\mathscr{T}} \wedge \underline{R}$ is contractible by Proposition 2.8. Since this is a ring object, it is contractible if and only if its fixed point spectrum is contractible. □

Proposition 2.14. The derived defect base of C_2-equivariant $K\mathbb{R}$-theory [Ati66] is $\mathscr{T}$, the trivial family.

Proof. We need to show that $K\mathbb{R}$ is $\mathscr{T}=\mathscr{P}$-nilpotent. In view of [MNN, Thm. 6.41], it suffices to show that the geometric fixed point spectrum $\Phi^{C_2}K\mathbb{R}=(\widetilde{E}\mathscr{P}\wedge R)^{C_2}$ is contractible. In this language the relevant calculation appears in [Faj95, Thm. 5.2] and in [HHR11, §7.3]; however the result follows from [Ati66, Prop. 3.2 and Lem. 3.7]. In fact, in the proof of [Faj95, Thm. 5.2], it is observed that the cube of the Euler class of the reduced regular representation of C_2 vanishes in $K\mathbb{R}$. □

Let MO and MU denote the genuine G-equivariant real and complex cobordism spectra of tom Dieck [tD70, BH72]. When $G=C_2$, let $M\mathbb{R}$ denote the real G-equivariant complex cobordism spectrum of Landweber [Lan68].

Proposition 2.15. The derived defect base of any of MO, MU, and $M\mathbb{R}$ is $\mathscr{A}\ell\ell$, the family of all subgroups of G.

Proof. We need to show that there is no proper family $\mathscr{F}$ such that any of these G-spectra is $\mathscr{F}$-nilpotent. By Corollary 2.5, to prove this for a G-ring spectrum R, it suffices to show that

$$0\neq \pi_*\Phi^G R\left(\cong \pi_*^G\widetilde{E}\mathscr{P}\wedge R\cong \pi_*^G R[e_{\bar{\rho}_G}^{-1}]\right).$$

In each of the stated cases this is known. The results for MO and MU are due to tom Dieck and can be found in [May96, §XV Lem. 3.1] and [tD70, Lem. 2.2] respectively. For $M\mathbb{R}$ this is [Lan68, Cor. 3.4]. □

2.3. **The class $\mathscr{F}^{\mathbf{Nil}}$ and the homotopy limit spectral sequence.** Before proving the equivalence of Conditions (1) and (3) from Theorem A in Theorem 2.25 below, we will give an alternate construction of $E\mathscr{F}$ and the $\mathscr{F}$-homotopy limit spectral sequence. These alternative constructions appear in the foundational work of Dress [Dre73], Greenlees [Gre92], and Greenlees-May [GM95b].

First, we describe another model for $E\mathscr{F}$. For a space Z, let $d_0\colon Z^{\bullet+1}\to *$ denote the standard augmented simplicial space which in degree n is the $(n+1)$-fold product of Z.

When $Z\neq\emptyset$, we can pick a point in Z to define a section s_{-1} of d_0. This section defines an additional degeneracy in each degree, or equivalently a retraction diagram of simplicial spaces

$$* \xrightarrow{s_{-1}} Z^{\bullet+1}\xrightarrow{d_0} * \tag{2.16}$$

with a simplicial homotopy $s_{-1}d_0\simeq \mathrm{Id}$ [GJ99, §III.5]. We will call an augmented simplicial space admitting extra degeneracies *split.*

When Z is a G-space, it is necessary and sufficient for Z to have a G-fixed point to split $Z^{\bullet+1}$ as a simplicial G-space. More generally, if $Z^H\neq\emptyset$ for $H\leq G$, then $\mathrm{Res}^G_H(Z^{\bullet+1})\simeq(\mathrm{Res}^G_H Z)^{\bullet+1}$ is split as a simplicial H-space. This implies that $G/H\times Z^{\bullet+1}\simeq \mathrm{Ind}_H^G\mathrm{Res}^G_H Z^{\bullet+1}$ is split as an augmented simplicial G-space.

Proposition 2.17 (cf. [GM95b, p. 119])**.** Let $\mathscr{F}$ be a family of subgroups of G and consider the G-space $X=\coprod_{H\in I}G/H$, where $I\subset\mathscr{F}$ contains a representative from each conjugacy class of maximal subgroups in $\mathscr{F}$.

Then there is an equivalence

$$|X^{\bullet+1}|\simeq E\mathscr{F}.$$

Moreover, if $H\in\mathscr{F}$ then $G/H\times X^{\bullet+1}$ is split.

Proof. This follows easily from the observations above and the characterization of $E\mathscr{F}$ from (2.6), since taking fixed points commutes with geometric realizations and products. In particular, $|X^{\bullet+1}|^H$ is contractible when X has an H-fixed point and is empty otherwise. □

The geometric realization of a simplicial G-space, $Z = |W_\bullet|$, admits two standard increasing filtrations by G-CW subcomplexes. The first is the filtration by dimension:

$$F_{-1}Z = \emptyset \subseteq F_0 Z \subseteq \cdots \subseteq F_\infty Z = Z$$

and depends on a choice of G-CW structure on Z. The second arises from the skeletal filtration on Z:

$$F'_{-1}Z = \emptyset \subseteq F'_0 Z \subseteq \cdots \subseteq F'_\infty Z = Z.$$

Here $F'_n Z := \operatorname{hocolim}_{\Delta^{\mathrm{op}}_{\leq n}} W_\bullet$ is the n-skeleton of Z and depends on the presentation of Z as the geometric realization of a simplicial G-space.

Fixing a G-spectrum M and applying $F(-_+, M)$ to these two filtrations, we obtain two towers of G-spectra, $\{F(F_n Z_+, M)\}_{n\geq 0}$ and $\{F(F'_n Z_+, M)\}_{n\geq 0}$. In general, if we apply π^G_* to a bounded below tower we obtain an exact couple and an associated spectral sequence conditionally converging to the homotopy groups of the homotopy inverse limit of the tower[4] [Boa99, §7].

In the case of the first tower, we are using a G-CW filtration on Z which satisfies

$$F_n Z / F_{n-1} Z \simeq \bigvee_{i\in I_n} G/H_{i_+} \wedge S^n,$$

where I_n is the set of orbits of n-cells of Z. The E_1-complex associated to the tower $\{F(F_n Z, M)\}_{n\geq 0}$ is

$$E_1^{s,t} = \pi^G_{t-s} F(F_s Z / F_{s-1} Z, M) \cong \prod_{i\in I_s} \pi^{H_i}_t M, \tag{2.18}$$

where the d_1-differential is induced by the attaching maps. This yields the equivariant analogue of the Atiyah-Hirzebruch spectral sequence whose E_2-term is, by definition, the *Bredon cohomology* of Z with coefficients in $\pi^{(-)}_* M$:

$$H^s_G(Z; \pi^{(-)}_t M) \Longrightarrow \pi^G_{t-s} F(Z_+, M). \tag{2.19}$$

If M is a G-ring spectrum then (2.19) is a spectral sequence of algebras [GM95b, App. B].

For the second tower we obtain the equivariant analogue of the Bousfield-Kan spectral sequence [Bou89, §3]:

$$\pi^s \pi^G_t(F(W_{\bullet +}; M)) \Longrightarrow \pi^G_{t-s} F(Z_+; M). \tag{2.20}$$

Here, the E_2-term is the cohomology of the graded cosimplicial abelian group $\pi^G_*(F(W_{\bullet +}, M))$. In Section 3 we will discuss the E_2-terms of these two spectral sequences further.

Proposition 2.21. Suppose that Z is the geometric realization of a simplicial G-space $W_\bullet$ and M is a G-spectrum. If W_n is discrete for each n, then the two spectral sequences (2.19) and (2.20) are isomorphic from the E_2-page on.

Proof. To compare (2.19) and (2.20) we would like a map between the associated towers. We do have an equivalence $F_\infty Z \simeq F'_\infty Z$, but in general this equivalence need not respect the filtrations. However, when $W_\bullet$ is degree-wise discrete, then for each n, $F'_n Z$ is n-dimensional and the skeletal filtration on Z is just the dimension filtration for a different choice of G-CW structure on Z. In this case, we can find an equivalence $s\colon F_\infty Z \to F'_\infty Z$ which respects the filtrations [May96, Cor. 3.5] and hence induces a map from the spectral sequence in (2.20) to the spectral sequence of (2.19). Applying the same argument to an inverse equivalence $t\colon F'_\infty Z \to F_\infty Z$ and to a homotopy $ts \simeq \mathrm{Id}$, we obtain a homotopy equivalence of E_1-complexes and hence an isomorphism at E_2. □

[4]Alternatively one can apply $\pi^{(-)}_\star$ to obtain a Mackey functor-valued, $RO(G)$-graded spectral sequence. This variant, although useful, will not be required for this paper.

We now turn our attention to $E\mathscr{F} = \operatorname{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} i$ for $i : \mathscr{O}(G)_{\mathscr{F}} \subset \mathcal{S}_G$ the inclusion. We will model this homotopy colimit as the geometric realization of the standard two-sided bar construction (see Appendix A.1 for further details):

$$E\mathscr{F} \simeq |B_\bullet(*, \mathscr{O}(G)_{\mathscr{F}}, i)|. \tag{2.22}$$

Definition 2.23. Let M be a G-spectrum. The *$\mathscr{F}$-homotopy limit spectral sequence associated to M* is the homotopy spectral spectral sequence associated to the tower

$$\{F(\mathrm{sk}_n E\mathscr{F}_+, M)\}_{n\geq 0},$$

where $E\mathscr{F}$ is equipped with the simplicial structure of (2.22).

Proposition 2.24. Let N be a G-spectrum. Then there is an isomorphism, from E_2 on, between:

(1) The $\mathscr{F}$-homotopy limit spectral sequence:

$$\pi^s\pi_t^G(F(B_\bullet(*, \mathscr{O}(G)_{\mathscr{F}}, i)_+, N)) \Longrightarrow \pi_{t-s}^G F(E\mathscr{F}_+, N) \cong \pi_{t-s}^G \operatorname*{holim}_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} F(G/H_+, N)$$

from Definition 2.23,

(2) the Bousfield-Kan spectral sequence

$$\pi^s\pi_t^G(F(X_+^{\bullet+1}, N)) \Longrightarrow \pi_{t-s}^G F(E\mathscr{F}_+, N) \cong \pi_{t-s}^G \operatorname*{holim}_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} F(G/H_+, N)$$

associated to a simplicial presentation of $E\mathscr{F}$ from Proposition 2.17, and

(3) the equivariant Atiyah-Hirzebruch spectral sequence

$$H_G^s(E\mathscr{F}; \pi_t^{(-)} N) \Longrightarrow \pi_{t-s}^G F(E\mathscr{F}_+, N) \cong \pi_{t-s}^G \operatorname*{holim}_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} F(G/H_+, N).$$

Moreover, when $N = F(Y, M)$, for two G-spectra Y and M such that M is $\mathscr{F}$-nilpotent, the above spectral sequences abut to $M_G^*(Y)$.[5]

Proof. When G is discrete, both $X^{\bullet+1}$ and $B_\bullet(*, \mathscr{O}(G)_{\mathscr{F}}, i)$ are degree-wise discrete. So it follows from Proposition 2.21 that all three spectral sequences are forms of the Atiyah-Hirzebruch spectral sequence for $E\mathscr{F}$ and hence isomorphic. The final claim follows from Proposition 2.12 and the isomorphism $\pi_{t-s}^G F(Y_+, M) \cong M_G^{s-t}(Y)$. $\square$

To proceed, we will need to recall some results on towers of G-spectra from [Mat15, §3]. We denote by $\mathrm{Tow}(\mathrm{Sp}_G) = \mathrm{Fun}((\mathbb{Z}_{\geq 0})^{\mathrm{op}}, \mathrm{Sp}_G)$ the ∞-category of towers in Sp_G. Inside this ∞-category is $\mathrm{Tow}^{\mathrm{nil}}(\mathrm{Sp}_G) \subset \mathrm{Tow}(\mathrm{Sp}_G)$, the full subcategory of nilpotent towers, i.e., those towers $\{X_n\}_{n\geq 0}$ such that for some $N \geq 0$ and all $k \geq 0$, the map $X_{N+k} \to X_k$ is zero. We denote by $\mathrm{Tow}^{\mathrm{fast}}(\mathrm{Sp}_G) \subset \mathrm{Tow}(\mathrm{Sp}_G)$ the full subcategory of *quickly converging* towers, i.e., those towers $\{X_n\}_{n\geq 0}$ such that the cofiber of the canonical map of towers $\{\operatorname{holim} X_n\} \to \{X_n\}_{n\geq 0}$ is contained in $\mathrm{Tow}^{\mathrm{nil}}(\mathrm{Sp}_G)$. It follows from the definitions that $\mathrm{Tow}^{\mathrm{fast}}(\mathrm{Sp}_G) \subset \mathrm{Tow}(\mathrm{Sp}_G)$ is a thick subcategory, and that exact endofunctors of Sp_G preserve $\mathrm{Tow}^{\mathrm{fast}}(\mathrm{Sp}_G)$.

We can now formulate the main result of this subsection, which in particular establishes the equivalences between Conditions (1) and (3) from Theorem A.

Theorem 2.25. The following three conditions on a G-spectrum M are equivalent:

(1) The G-spectrum M is $\mathscr{F}$-nilpotent.

(2) The restriction map $\mathrm{Res}_{\mathscr{F}}^G : M \longrightarrow \operatorname{holim}_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} F(G/H_+, M) \simeq F(E\mathscr{F}_+, M)$ is an equivalence and the associated tower $\{F(\mathrm{sk}_n E\mathscr{F}_+, M)\}_{n\geq 0}$ converges quickly.

[5]As a consequence of Theorem 2.25, Item 3 below, they actually converge strongly to their abutment.

(3) The map $M \to F(E\mathscr{F}_+, M)$ is an equivalence and there are integers m and $n \geq 2$ such that for every G-spectrum Y, the $\mathscr{F}$-homotopy limit spectral sequence:

$$E_2^{s,t} = H^s(E\mathscr{F}; \pi_t^{(-)} F(Y, M)) \Longrightarrow M_G^{s-t}(Y)$$

has a horizontal vanishing line of height m on the E_n-page. In other words, $E_k^{s,*} = 0$ for all $s > m$ and $k \geq n$.

Proof. The equivalence (2) $\iff$ (3) is [Mat15, Prop. 3.12] combined with the identification of the $\mathscr{F}$-spectral sequence from Proposition 2.24.

We will now show (1) $\iff$ (2). Let $A_{\mathscr{F}} = \prod_{H \in \mathscr{F}} F(G/H_+, S)$, so that a G-spectrum is $\mathscr{F}$-nilpotent if and only if it is $A_{\mathscr{F}}$-nilpotent. Write $E\mathscr{F} = |X^{\bullet+1}|$ for $X = \coprod_{H \in \mathscr{F}} G/H$. Then the tower $\{F(\mathrm{sk}_n E\mathscr{F}_+, M\})$ is the Tot tower of the $A_{\mathscr{F}}$-cobar complex of M. This is a quickly converging tower with homotopy limit M if and only if the $A_{\mathscr{F}}$-Adams tower [MNN, Construction 2.2] is nilpotent (note that the $A_{\mathscr{F}}$-Adams tower is the cofiber of the map of towers $\{M\} \to \{F(\mathrm{sk}_n E\mathscr{F}_+, M)\}$ by [MNN, Prop. 2.14]). Furthermore, that holds if and only if and M is $A_{\mathscr{F}}$-nilpotent [MNN, Prop. 4.7]. □

Recall also that we can quantify nilpotence, leading to the notion of the *$\mathscr{F}$-exponent* of an $\mathscr{F}$-nilpotent G-spectrum M, denoted $\exp_{\mathscr{F}}(M)$ [MNN, Def. 6.36]. Recall again that, associated to G and $\mathscr{F}$, there is the commutative algebra $A_{\mathscr{F}} := \prod_{H \in \mathscr{F}} F(G/H_+, S)$ in Sp_G. The fiber I of the canonical map $S \to A_{\mathscr{F}}$ is a non-unital algebra, and the $\mathscr{F}$-exponent of $M \in \mathscr{F}^{\mathrm{Nil}}$ is the minimum number $n \geq 0$ such that $(I^{\wedge n} \to S) \wedge M$ is zero. For $Y \in \mathrm{Sp}_G$, we will denote by $E_*^{*,*}(Y)$ the $\mathscr{F}$-homotopy limit spectral sequence abutting to $M_G^*(Y)$. We can then formulate the following alternate descriptions of the $\mathscr{F}$-exponent.

Proposition 2.26. For a nontrivial $\mathscr{F}$-nilpotent spectrum M, the following integers are equal:

- the $\mathscr{F}$-exponent $\exp_{\mathscr{F}}(M)$,
- the minimal n such that the canonical map $M \simeq F(E\mathscr{F}_+, M) \longrightarrow F(\mathrm{sk}_{n-1} E\mathscr{F}_+, M)$ in Sp_G admits a retraction,
- the minimal n' such that M is a retract of an $F(Z_+, M)$ for an $(n'-1)$-dimensional G-CW complex Z with isotropy in $\mathscr{F}$,
- and the minimum $s \geq 0$ such that for all $Y \in \mathrm{Sp}_G$ and $k \geq s$, $E_{s+1}^{k,*}(Y) = E_\infty^{k,*}(Y) = 0$.

Proof. This follows easily from results in [MNN]. Fix the G-space $X := \coprod_{H \in \mathscr{F}} G/H$ and the associated simplicial G-space $X^{\bullet+1}$ which realizes to $E\mathscr{F}$. One sees that the identification $A_{\mathscr{F}} \simeq F(X_+, S)$ generalizes to an identification of cosimplicial commutative algebras in Sp_G, namely the *cobar construction* $\mathrm{CB}^\bullet(A_{\mathscr{F}})$ (cf. [MNN, Sec. 2.1]) is equivalent to $F(X_+^{\bullet+1}, S)$. In view of this, the equality of the first two integers follows from [MNN, Prop. 4.9]. To compare n' and n we first note that by setting $Z = \mathrm{sk}_{n-1} E\mathscr{F}$ we see that $n' \leq n$. The other inequality follows because $F(Z_+, M)$, for an $(n'-1)$-dimensional G-CW complex Z with isotropy in $\mathscr{F}$ and for any G-spectrum M, has $\mathscr{F}$-exponent $\leq n'$.

Finally, we show $n = s$. Using $\mathrm{CB}^\bullet(A_{\mathscr{F}}) \simeq F(X_+^{\bullet+1}, S)$ again, one sees that our $\mathscr{F}$-homotopy limit spectral sequence can be identified with the $A_{\mathscr{F}}$-based Adams spectral sequence as in [Gre92], and it is well-known that the Adams filtration of a map $f \colon \Sigma^{-*} Y \to M$ in $M_G^*(Y)$ is exactly the maximum q such that f factors through $I^{\wedge q} \wedge M \to M$. It follows that $(n-1)$ is the (precisely) maximum $A_{\mathscr{F}}$-Adams filtration of any map into M, which implies that $E_\infty^{*,k}(Y) = 0$ for $k \geq n$ and for any G-spectrum Y; moreover, n is minimal with respect to this property.

It remains to show that the $\mathscr{F}$-spectral sequence degenerates at E_{n+1}, or equivalently that $d_i = 0$ for $i \geq n+1$. This is a very general assertion about these types of generalized Adams spectral sequences. For simplicity of notation, we assume that $Y = S^0$. The $E_1^{p,q}$-page of the spectral sequence gives the homotopy groups $\pi_p(\mathrm{fib}\,(\mathrm{Tot}_q \to \mathrm{Tot}_{q-1}))$ for the cosimplicial object $M \otimes \mathrm{CB}^\bullet(A_{\mathscr{F}})$. By [MNN, Prop. 2.14], we have

$$\mathrm{fib}\,(\mathrm{Tot}_q \to \mathrm{Tot}_{q-1}) = \mathrm{cofib}(I^{\wedge q+1} \to I^{\wedge q}) \wedge M = I^{\wedge q}/I^{\wedge q+1} \wedge M.$$

If a class survives to E_{n+1}, then it can be lifted to

$$\mathrm{fib}\,(\mathrm{Tot}_{q+n} \to \mathrm{Tot}_{q-1}) = I^{\wedge q}/I^{\wedge q+n+1} \wedge M,$$

by [MNN, Prop. 2.14] again. Consider now the diagram

$$\begin{array}{ccccc} & & I^{\wedge q}/I^{\wedge q+n+1} \wedge M & \longrightarrow & \Sigma I^{\wedge q+n+1} \wedge M\ . \\ & \swarrow\!\!{}^{\text{- -}} & \downarrow \psi & & \downarrow \phi \\ I^{\wedge q} \wedge M & \longrightarrow & I^{\wedge q}/I^{\wedge q+1} \wedge M & \xrightarrow{\partial} & \Sigma I^{\wedge q+1} \wedge M. \end{array}$$

We claim that, under the hypotheses, there exists a dotted arrow making the diagram commute. Therefore, our class can be in fact lifted to $\mathrm{fib}(\mathrm{Tot} \to \mathrm{Tot}_{q-1})$ and so is a permanent cycle in the $\mathscr{F}$-spectral sequence. To see this, we need to argue that the composite $\partial \circ \psi$ is null-homotopic. However, this follows from the fact that the diagram commutes and that ϕ is null-homotopic by hypothesis on M. □

The proof of Theorem A is now complete except for the identification of the E_2-term of the homotopy limit spectral sequence, and this will be completed in Section 3.1.

Remark 2.27. One can dualize [Mat15, §3] since the notion of a stable ∞-category is self-dual. We thus obtain inside $\mathrm{Fun}(\mathbb{Z}_{\geq 0}, \mathrm{Sp}_G)$ the nilpotent and quickly converging directed systems. The latter subcategory is thick and stable under exact endofunctors of Sp_G. The exact couples associated to such directed systems once again define homological-type spectral sequences with horizontal vanishing lines. For example, when M is $\mathscr{F}$-nilpotent, $\{\mathrm{sk}_n E\mathscr{F}_+ \wedge M\}_{n\geq 0}$ is a quickly converging directed system. It follows that for arbitrary $X \in \mathrm{Sp}_G$, the $\mathscr{F}$-homotopy colimit spectral sequence

$$E^2_{s,t} = H^G_s(E\mathscr{F}; \pi^{(-)}_t F(X, M)) \cong \operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} {}_s\, M^H_t(X) \Longrightarrow M^G_{t+s}(X)$$

has a horizontal vanishing line at a finite page.

Coupling this with the analogous result for the homotopy limit spectral sequence forces the generalized $\mathscr{F}$-Tate spectral sequence of [GM95b, §22] to collapse to zero at some finite stage. Indeed, the positive degree terms of this spectral sequence are a quotient of the positive degree terms in the $\mathscr{F}$-homotopy limit spectral sequence while the terms in degrees less than -1 are a subset of the positive degree terms in the $\mathscr{F}$-homotopy colimit spectral sequence (cf. (3.11)). Our vanishing results now imply the collapse of the $\mathscr{F}$-Tate spectral sequence at a finite stage. By Proposition 2.8 this spectral sequence converges to 0.

3. Analysis of the spectral sequences

Let G be a finite group and $\mathscr{F}$ a family of subgroups. Let $X = \coprod_{H\in\mathscr{F}} G/H$ be as in Proposition 2.17. As observed in the previous section, the $\mathscr{F}$-homotopy limit spectral sequence can be viewed as the Bousfield-Kan spectral sequence [BK72, Ch. X] associated to the cosimplicial G-spectrum $F(X^{\bullet+1}_+, M)$ or as an equivariant Atiyah-Hirzebruch spectral sequence with E_2-term

$$H^*_G(|X^{\bullet+1}|_+; \pi^{(-)}_* M) \cong H^*_G(E\mathscr{F}_+; \pi^{(-)}_* M).$$

In Section 3.1 we recall that this E_2-term can be identified with the derived functors $\lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi^{(-)}_* M$ as expected.

There is also an $\mathscr{F}$-homotopy colimit spectral sequence and the chain complexes calculating the E_2-terms of the $\mathscr{F}$-homotopy colimit and limit spectral sequences can be glued together to form the associated Amitsur-Dress-Tate cohomology groups $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}M)$. In Section 3.2 we will review this construction and recall a few vanishing results. These results play a critical role in the proofs of Theorems B and C and Corollary 1.4 in Section 3.3. They will also be used in the proof of the generalized hyperelementary induction theorem Proposition 4.12, in Section 4. We conclude this section with a form of Quillen's stratification theorem (Proposition 3.27).

3.1. **Bredon (co)homology and derived functors.** Let $\mathscr{C}$ be a small category and $\mathbb{Z}\mathscr{C}$ the category of contravariant functors from $\mathscr{C}$ to abelian groups; $\mathbb{Z}\mathscr{C}$ is an abelian category with kernels and cokernels calculated object-wise. The category $\mathbb{Z}\mathscr{C}$ also has enough projectives: by the Yoneda lemma, one sees that the functors $\mathbb{Z}\{\mathscr{C}(-,c)\}$ for $c \in \mathscr{C}$ form a set of projective generators.

For any pair M_1 and M_2 of objects in $\mathbb{Z}\mathscr{C}$ we can, as usual, take a projective resolution $P_\bullet$ of M_1 to calculate the Ext-groups:

$$\mathrm{Ext}^*_{\mathbb{Z}\mathscr{C}}(M_1, M_2) \cong H^*(\mathbb{Z}\mathscr{C}(P_\bullet, M_2)).$$

Now let $\underline{\mathbb{Z}}$ denote the constant functor $c \mapsto \mathbb{Z}$. This functor corepresents the (covariant, left exact) limit functor, i.e., for any $M \in \mathbb{Z}\mathscr{C}$, we have

$$\mathbb{Z}\mathscr{C}(\underline{\mathbb{Z}}, M) \cong \lim_{\mathscr{C}^{\mathrm{op}}} M,$$

and we obtain the derived functors of $\lim_{\mathscr{C}^{\mathrm{op}}}$:

$$\lim_{\mathscr{C}^{\mathrm{op}}}{}^*(M) \cong \mathrm{Ext}^*_{\mathbb{Z}\mathscr{C}}(\underline{\mathbb{Z}}, M). \tag{3.1}$$

We now specialize to the primary case of interest for us.

Definition 3.2. The category of *coefficient systems* (on a finite group G) is the category $\mathbb{Z}\mathscr{O}(G)$ of contravariant functors from $\mathscr{O}(G)$ to abelian groups.

Examples 3.3. (1) Associated to any G-set X we obtain a coefficient system $\mathbb{Z}[X]$ defined by

$$\mathbb{Z}[X]\colon G/H \mapsto \mathbb{Z}\{\mathrm{Ho}\,\mathcal{S}_G(G/H, X)\} \cong \mathbb{Z}[X^H].$$

When $X = G/H$, $\mathbb{Z}[X]$ is the projective functor $\mathbb{Z}\{\mathscr{O}(G)(-, G/H)\}$ considered above. More generally, by decomposing X into orbits we see that $\mathbb{Z}[X]$ is a direct sum of projective functors.

(2) Let X be a G-CW complex and for each $n \geq 0$ let X_n be the G-set of n-cells in X. The attaching maps define a chain complex of coefficient systems $C_*(X) := \mathbb{Z}[X_*]$.

(3) Let $\underline{\mathbb{Z}[\mathscr{F}]}$ denote the coefficient system

$$\underline{\mathbb{Z}[\mathscr{F}]}\colon G/K \mapsto H_*(E\mathscr{F}^K; \mathbb{Z}) = H_0(E\mathscr{F}^K; \mathbb{Z}).$$

By (2.6), we see that $\underline{\mathbb{Z}[\mathscr{F}]}(G/K) = \mathbb{Z}$, when $K \in \mathscr{F}$, and is zero otherwise.

(4) A G-spectrum M defines a graded coefficient system $\pi_*^{(-)}M$ by

$$\pi_*^{(-)}M\colon G/H \mapsto \pi_*^G F(G/H_+, M) \cong \pi_*^H M.$$

Lemma 3.4. Let $\mathscr{C} \subset \mathscr{D}$ be the inclusion of a full subcategory. Suppose for any $d \in \mathscr{D}$ such that $d \notin \mathscr{C}$, the category $\mathscr{C}_{d/}$ is empty. Let $\underline{\mathbb{Z}}_{\mathscr{C}} \in \mathbb{Z}\mathscr{D}$ denote the functor

$$\underline{\mathbb{Z}}_{\mathscr{C}}(d) = \begin{cases} \mathbb{Z} & d \in \mathscr{C} \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $M \in \mathbb{Z}\mathscr{D}$, there is a functorial isomorphism

$$\mathrm{Ext}^s_{\mathbb{Z}\mathscr{D}}(\underline{\mathbb{Z}}_{\mathscr{C}}, M) \simeq \lim_{\mathscr{C}^{op}}{}^s (M|_{\mathscr{C}}).$$

Proof. The object $\underline{\mathbb{Z}}_{\mathscr{C}}$ is the homotopical left Kan extension of $\underline{\mathbb{Z}} \in \mathbb{Z}\mathscr{C}$ to $\mathbb{Z}\mathscr{D}$. The statement reduces now to the adjunction isomorphism at the level of homotopy categories. □

Proposition 3.5. Let $C \in \mathbb{Z}\mathcal{O}(G)$ be a coefficient system. Then there is an identification between the Bredon cohomology $H^s_G(E\mathscr{F}; C)$ and the derived functors $\lim^s_{\mathscr{O}(G)^{op}_{\mathscr{F}}} C$.

Proof. The Bredon cohomology groups as defined previously are the cohomology groups of the cochain complex

$$\mathbb{Z}\mathscr{O}(G)(C_*(E\mathscr{F}), C).$$

Now the desired isomorphism follows from Lemma 3.4 applied to $\mathscr{O}(G)_{\mathscr{F}} \subset \mathscr{O}(G)$ and the fact that $H_*(E\mathscr{F}^K; \mathbb{Z})$ is always concentrated in degree 0; hence $C_*(E\mathscr{F}) \to H_0(E\mathscr{F}^{(-)}; \mathbb{Z})$ is a projective resolution of $\mathbb{Z}[\mathscr{F}]$.

□

Corollary 3.6. Fix a G-spectrum M. Let $E_2^{s,t}$ denote the E_2-term of the $\mathscr{F}$-homotopy limit spectral sequence. Then there is a chain of isomorphisms:

$$\begin{aligned} E_2^{s,t} &\cong H^s_G(E\mathscr{F}; \pi_t^{(-)} M) \\ &\cong \mathrm{Ext}^{s,t}_{\mathbb{Z}\mathscr{O}(G)}(\underline{\mathbb{Z}[\mathscr{F}]}, \pi_*^{(-)} M) \\ &\cong \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}}{}^s \pi_t^H M \\ &\cong \mathrm{Ext}^{s,t}_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}(\underline{\mathbb{Z}}, \pi_*^{(-)} M). \end{aligned}$$

In particular, the 0-line is $\lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi_*^H M$.

Proof. The identification of the E_2-term as the derived functors of the limit is due to Bousfield and Kan [BK72, Ch. XI] and the remaining isomorphisms follow from the above discussion. □

The above results and identifications dualize, although we leave the details to the reader: A G-spectrum M defines a *covariant* functor $\pi_*^{(-)} M$ from $\mathscr{O}(G)$ to (graded) abelian groups by

$$(\pi_*^{(-)} M)(G/H) = \pi_*^G(G/H_+ \wedge M) \cong \pi_*^H M.$$

Now the skeletal filtration on $E\mathscr{F}$ defines a homological Atiyah-Hirzebruch spectral sequence with the following E^2-identifications (cf. [May96, §1.4]):

$$E^2_{s,t} \cong H^G_s(E\mathscr{F}; \pi_t^{(-)} M) \cong \mathrm{Tor}^{\mathbb{Z}\mathscr{O}(G)}_{s,t}(\underline{\mathbb{Z}[\mathscr{F}]}, \pi_*^{(-)} M) \cong \mathrm{Tor}^{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}_{s,t}(\underline{\mathbb{Z}}, \pi_*^{(-)} M)$$

$$\cong \operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}}{}_s \pi_t^H M \Longrightarrow \pi^G_{t+s} \operatorname*{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} G/H_+ \wedge M.$$

Here, for a G-space X, the Bredon homology $H^G_*(X; \pi_t^{(-)} M)$ is defined to be the homology of the chain complex

$$C^G_*(X; \pi_t^{(-)} M) := C_*(X) \otimes_{\mathbb{Z}\mathscr{O}(G)} \pi_t^{(-)} M$$

formed from the tensor product of graded functors.

3.2. **Amitsur-Dress-Tate cohomology.** Let $C \in \mathbb{Z}\mathcal{O}(G)$ and consider the Bredon cohomology $H^s_G(E\mathscr{F}; C) = \lim^s{}_{\mathcal{O}(G)^{op}_{\mathscr{F}}} C$. When C comes from a *Mackey functor* on G (e.g., as the homotopy groups of a G-spectrum), these groups are forced to be $|G|$-torsion for $s > 0$. This will be fundamental for our computational applications of $\mathscr{F}$-nilpotence.

In this subsection, we will review some of the theory of Amitsur-Dress-Tate cohomology [GM95b, §21], which we will use to prove these results. For notational simplicity, we will always assume that our Mackey functor is given to us as the homotopy groups of a G-spectrum M.

Construction 3.7. We can splice together the E_1-pages of the homological and cohomological spectral sequences from the previous section to define Amitsur-Dress-Tate cohomology.

For this purpose let $C_*(E\mathscr{F}; \pi_*^{(-)}M)$ and $C^*(E\mathscr{F}; \pi_*^{(-)}M)$ denote the Bredon cellular chains and cochain complexes on $E\mathscr{F}$ with coefficients in $\pi_*^{(-)}M$. These complexes have degree zero (co)homology given by $\operatorname{colim}_{\mathcal{O}(G)_{\mathscr{F}}} \pi_*^H M$ and $\lim_{\mathcal{O}(G)^{op}_{\mathscr{F}}} \pi_*^H M$, respectively, and we obtain a natural norm map (cf. (1.1))

$$\mathrm{N}\colon \operatorname*{colim}_{\mathcal{O}(G)_{\mathscr{F}}} \pi_*^H M \to \lim_{\mathcal{O}(G)^{op}_{\mathscr{F}}} \pi_*^H M. \tag{3.8}$$

As a result, we obtain a map of complexes

$$C_*(E\mathscr{F}; \pi_*^{(-)}M) \to C^*(E\mathscr{F}; \pi_*^{(-)}M) \tag{3.9}$$

determined by the condition that it induce (3.8) in π_0. We define the *Amitsur-Dress-Tate complex* $\widehat{C}^*(E\mathscr{F}; \pi_*^{(-)}M))$ to be the cofiber of the above map.

Definition 3.10 ([GM95b, Def. 21.1])**.** The *Amitsur-Dress-Tate cohomology* groups of $\mathscr{F}$ with coefficients in $\pi_*^{(-)}M$ are defined by

$$\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}M) := H^*(\widehat{C}^*(E\mathscr{F}; \pi_*^{(-)}M)).$$

We immediately obtain the following identification of the Amitsur-Dress-Tate cohomology in terms of (3.8):

$$\widehat{H}^s_{\mathscr{F}}(\pi_*^{(-)}M) \cong \begin{cases} H^{s,*}_G(E\mathscr{F}; \pi_*^{(-)}M) & \text{if } s > 0 \\ H^G_{-s-1,*}(E\mathscr{F}; \pi_*^{(-)}M) & \text{if } s < -1 \\ \operatorname{coker} \mathrm{N} & \text{if } s = 0 \\ \ker \mathrm{N} & \text{if } s = -1 \end{cases} \tag{3.11}$$

We will now record some basic properties of Amitsur-Dress-Tate cohomology.

Proposition 3.12. Suppose that R is a G-ring spectrum and M is an R-module, then:

1. The Amitsur-Dress-Tate cohomology groups $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}R)$ have an induced graded $\pi_*^G R$-algebra structure and $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}M)$ is a graded module over $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}R)$ such that the isomorphisms in (3.11) respect this structure.
2. If $x = \operatorname{Ind}_H^G y \in \pi_*^G R$ for some $H \in \mathscr{F}$ and $y \in \pi_*^H R$, then $x \cdot \widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}R) = 0$.
3. The commutative ring $\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)}S)$ is annihilated by $|G|$. We let $n(\mathscr{F})$ be the minimal positive integer which vanishes in $\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)}S)$, so that $n(\mathscr{F}) \mid |G|$.
4. The number $n(\mathscr{F})$ from (3) is the minimal positive integer n such that $n \cdot \widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}M) = 0$, for *every* R and M.

In particular, if $i > 0$ then $H_i(E\mathscr{F}; \pi_*^{(-)}M)$ and $H^i(E\mathscr{F}; \pi_*^{(-)}M)$ are $n(\mathscr{F})$-torsion.

Family $\mathscr{F}$	$n(\mathscr{F})$
$\mathscr{T}$	60
$\mathscr{C}_{(2)}$	30
$\mathscr{A}_{(2)}$	15
$\mathscr{C}_{(3)} = \mathscr{A}_{(3)}$	20
$\mathscr{C}_{(5)} = \mathscr{A}_{(5)}$	12
$\mathscr{C}$	2
$\underline{\mathscr{A}\ell\ell} = \mathscr{A} = \mathscr{A}^2 = \mathscr{E}$	6
$\mathscr{P}$	1
$\mathscr{A}\ell\ell$	1

(A) Indices of families of subgroups of A_5.

	$[A_5/e]$	$[A_5/C_2]$	$[A_5/C_3]$	$[A_5/C_2 \times C_2]$	$[A_5/C_5]$	$[A_5/\Sigma_3]$	$[A_5/D_{10}]$	$[A_5/A_4]$	$[A_5/A_5]$
e	60	30	20	15	12	10	6	5	1
C_2	0	2	0	3	0	2	2	1	1
C_3	0	0	2	0	0	1	0	2	1
$C_2 \times C_2$	0	0	0	3	0	0	0	1	1
C_5	0	0	0	0	2	0	1	0	1
Σ_3	0	0	0	0	0	1	0	0	1
D_{10}	0	0	0	0	0	0	1	0	1
A_4	0	0	0	0	0	0	0	1	1
A_5	0	0	0	0	0	0	0	0	1

(B) Table of marks for A_5.

FIGURE 3.18

Definition 3.13. For a finite group G and a family $\mathscr{F}$ of subgroups of G, the integer $n(\mathscr{F})$ in Proposition 3.12, (3) is called *the index of the family* $\mathscr{F}$ (of subgroups of G).

Proof of Proposition 3.12. The argument below is an algebraic one. Although our applications concern G-spectra, their appearance in this proposition is a red herring; it could just as easily be stated in terms of general Green and Mackey functors.

The first claim is a graded form of [Dre73, Prop. 2.3]. It follows that $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}R)$ is a module over

$$\widehat{H}^0_{\mathscr{F}}(\pi_*^{(-)}R) \cong \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi_*^{(-)}R/\mathrm{Im}\left(\mathrm{Ind}^G_{\mathscr{F}}\right) = \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi_*^{(-)}R/(\sum_{H\in\mathscr{F}} \mathrm{Im}\,\mathrm{Ind}^G_H(\pi_*^H R)). \tag{3.14}$$

This immediately implies the second claim. The fourth claim is clear because every $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}M)$ is a module over $\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)}S)$, and the third claim will be addressed in the lemma below. □

Recall that we have $\pi_0^G S \simeq A(G)$, the Burnside ring of G. Jointly with (3.14) applied to $R = S$ and $* = 0$, this yields a description of the commutative ring $\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)}S)$ in terms of the Burnside rings $A(H)$ for certain subgroups $H \leq G$, and shows that claim (3) of Proposition 3.12 is equivalent to the following result.

Lemma 3.15. There is a minimal positive integer $n(\mathscr{F})$ such that there exists $x \in \mathrm{Im}\,\mathrm{Ind}^G_{\mathscr{F}} \subseteq A(G)$ and

$$y \in \ker\left(A(G) \xrightarrow{\mathrm{Res}^G_{\mathscr{F}}} \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} A(-)\right)$$

such that $n(\mathscr{F}) = x + y$. Furthermore, the integer $n(\mathscr{F})$ divides the group order $|G|$.

Proof. See [GM95b, Prop. 21.3 and Cor. 21.4]. □

Remark 3.16. The existence proof of $n(\mathscr{F})$ is constructive. In fact, computing $n(\mathscr{F})$ is a linear algebra problem involving the table of marks of G which can be calculated by a computer algebra package such as GAP.

Examples 3.17.
(1) When $G = A_5$ we have calculated the indices of various families in Figure 3.18A using the table of marks in Figure 3.18B.
(2) A prime p divides $n(\mathscr{P})$ if and only if there is a nontrivial homomorphism $G \to C_p$ or, equivalently, $H^1(BG; \mathbb{F}_p) \neq 0$ [GM95b, Ex. 21.5.(iii)]. In particular, if G is perfect, then we have $n(\mathscr{P}) = 1$.

3.3. **Artin induction and $\mathcal{N}$-isomorphism theorems.** Proposition 3.12 immediately implies the following more precise form of Theorem B:

Theorem 3.19. Let M and X be G-spectra and $\mathscr{F}$ a family of subgroups such that M is $\mathscr{F}$-nilpotent. Then each of the following maps

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} M_H^*(X) \xrightarrow{\operatorname{Ind}_{\mathscr{F}}^G} M_G^*(X) \xrightarrow{\operatorname{Res}_{\mathscr{F}}^G} \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M_H^*(X) \tag{3.20}$$

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} M^H_*(X) \xrightarrow{\operatorname{Ind}_{\mathscr{F}}^G} M^G_*(X) \xrightarrow{\operatorname{Res}_{\mathscr{F}}^G} \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M^H_*(X) \tag{3.21}$$

is an isomorphism after inverting $n(\mathscr{F})$, the index of the family $\mathscr{F}$.

Proof. Since $\mathscr{F}^{\mathrm{Nil}}$ is closed under tensors and cotensors, it suffices to prove the theorem in the case $X = S^0$. Since $\pi_*^{(-)} M = M_*^{(-)}(S^0) = M^{-*}_{(-)}(S^0)$, we see that (3.20) and (3.21) both reduce to statements about homotopy groups.

Set $n = n(\mathscr{F})$. Since M is $\mathscr{F}$-nilpotent, the $\mathscr{F}$-homotopy limit spectral sequence converges strongly and has a horizontal vanishing line, say of height m at the Nth page. We will now analyze the following composition of maps

$$\ker \operatorname{Res}_{\mathscr{F}}^G \hookrightarrow \pi_*^G M \twoheadrightarrow E_\infty^{0,*} \hookrightarrow E_2^{0,*} = \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} \pi_*^H M$$

where the composition of the latter two maps is $\operatorname{Res}_{\mathscr{F}}^G$. Now $\ker \operatorname{Res}_{\mathscr{F}}^G$ consists of those elements in $\pi_*^G M$ detected in positive filtration. The associated graded of this filtration on $\pi_*^G M$ is $\bigoplus_{s\geq 0} E_\infty^{s,*}$. These groups are n-torsion for $s > 0$ by Proposition 3.12 and 0 for $s > m$. So if x is detected in $E_2^{s,*}$ for $s > 0$ then nx is zero *modulo higher filtration.* Since the groups in filtration degree greater than m are zero we see that $n^m \cdot \ker \operatorname{Res}_{\mathscr{F}}^G = 0$. So $\operatorname{Res}_{\mathscr{F}}^G$ is an injection after inverting n.

Now suppose that $x \in E_2^{0,*}$ is not in the image of $\operatorname{Res}_{\mathscr{F}}^G$. Since the spectral sequence converges strongly x must support a differential. Suppose that $d_2 x = y \neq 0$. Since y is in positive filtration it is n-torsion and hence $d_2(nx) = ny = 0$. So nx survives to E_3. Inductively we see that $n^k x$ survives to the E_{2+k} page. Using the horizontal vanishing line we see that there must be a fixed k such that $n^k E_2^{0,*} \subset E_\infty^{0,*} = \operatorname{Im}\operatorname{Res}_{\mathscr{F}}^G$. It follows that $\operatorname{Res}_{\mathscr{F}}^G$ is an isomorphism after inverting n.

The claim for $\operatorname{Ind}_{\mathscr{F}}^G$ is easier and only requires that the map $\operatorname{Ind}_{\mathscr{F}}^G : E\mathscr{F}_+ \wedge M \to M$ be an equivalence, i.e., that M be $\mathscr{F}$-torsion, rather than that M be actually $\mathscr{F}$-nilpotent. Since inverting n commutes with homotopy colimits and ordinary colimits, if we tensor the $\mathscr{F}$-homotopy colimit spectral sequence

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}}{}_s (\pi_*^H M) \Longrightarrow \pi_*^G M$$

with $\mathbb{Z}[n^{-1}]$ we obtain the homotopy colimit spectral sequence for $M[n^{-1}]$. This spectral sequence collapses at E_2 onto the zero line by Proposition 3.12 and the claim for $\operatorname{Ind}_{\mathscr{F}}^G$ follows. $\square$

Theorem 3.22. Let R be an $\mathscr{F}$-nilpotent G-ring spectrum and let X be a G-space. Suppose further that for each $H \in \mathscr{F}$, the graded ring $R_H^*(X)$ is graded-commutative. Then the canonical map

$$\operatorname{Res}_{\mathscr{F}}^G : R_G^*(X) \longrightarrow \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^*(X)$$

is a uniform $\mathcal{N}$-isomorphism, i.e., there are positive integers K and L such that if $x \in \ker \operatorname{Res}_{\mathscr{F}}^G$ and $y \in \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} R_H^*(X)$ then $x^K = 0$ and $y^L \in \operatorname{Im} \operatorname{Res}_{\mathscr{F}}^G$. Moreover, after localizing at a prime p, $\operatorname{Res}_{\mathscr{F}}^G$ is a uniform $\mathcal{F}_p$-isomorphism.

Proof. Suppose that $x \in \ker \mathrm{Res}^G_{\mathscr{F}}$. It follows from the strong convergence of the $\mathscr{F}$-homotopy limit spectral sequence

$$E_2^{s,-t} = \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{F}}}{}^{s} R^t_H(X) \Longrightarrow R^{t+s}_G(X) \tag{3.23}$$

that x is detected in positive filtration. This spectral sequence has a horizontal vanishing line at the E_∞-page. More precisely, we know that for $K = \exp_{\mathscr{F}}(R)$, $E_\infty^{s,*} = 0$ when $s \geq K$. It follows that $x^K = 0$.

Now suppose that $y \in \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{F}}} R^*_H(X)$ is not in the image of $\mathrm{Res}^G_{\mathscr{F}}$. Convergence of the $\mathscr{F}$-homotopy limit spectral sequence implies that such an element must support a nontrivial differential, say $d_n(y) = z \neq 0$. Since z is in positive filtration, it is $N = n(\mathscr{F})$-torsion by Proposition 3.12. Replacing y with its square if necessary, we can assume that y is in even degrees. Now since $R^*_H(X)$ is a graded-commutative ring functorially in $H \in \mathscr{F}$, $\lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{F}}} R^*_H(X)$ is a graded-commutative ring. It now follows from the Leibniz rule that $d_n(y^N) = Ny^{N-1}z = 0$ and that y^N survives to the E_{n+1}-page. We can now argue by induction and, since the spectral sequence collapses at the E_{K+1}-page, it follows that for every $y \in \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{F}}} R^*_H(X)$, $y^{2N^{K-1}}$ survives the spectral sequence. Setting $L = 2N^{K-1}$ we that $\mathrm{Res}^G_{\mathscr{F}}$ is a uniform $\mathcal{N}$-isomorphism as described above.

To see that $\mathbb{Z}_{(p)} \otimes \mathrm{Res}^G_{\mathscr{F}}$ is a uniform $\mathcal{F}_p$-isomorphism we consider the p-localization of the $\mathscr{F}$-homotopy limit spectral sequence in (3.23). In general, p-localization does not commute with limits or homotopy inverse limits, but since p-localization is topologically realized by smashing with $S_{(p)}$ and the category of quickly converging towers is closed under smashing with constant towers, we see that the p-localization of (3.23) is the $\mathscr{F}$-homotopy limit spectral sequence for $F(X_+, R)_{(p)}$. Now the same argument above shows that if $x \in \ker \mathrm{Res}^G_{\mathscr{F}}$ then $x^K = 0$. Since the prime-to-p torsion in positive filtration in (3.23) is 0 after p-localization, we see that if $y \in \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{F}}} R^*_H(X)_{(p)}$, then $y^{2p^{k(M-2)}}$ survives the spectral sequence for some k. ☐

Remark 3.24. The horizontal vanishing line in fact implies $\ker(\mathrm{Res}^G_{\mathscr{F}})^{\exp_{\mathscr{F}}(R)} = 0$.

We conclude this section with several applications of Theorem 3.22 including Proposition 3.27, a form of Quillen's stratification theorem. First we will prove the following two elementary propositions, which were known to Quillen, which demonstrate how Theorem 3.22 implies Corollary 1.4.

Proposition 3.25. If $f\colon A \to B$ is an $\mathcal{N}$-isomorphism of commutative rings, then $\mathrm{Spec}(f)$ is a homeomorphism.

Proof. We factor f as $A \to A/\ker(f) \to B$. The first map induces a homeomorphism on Spec by [AM69, §1, ex. 21.iv)] and the second one a closed continuous surjection by [AM69, §1, ex. 21, v) and §5, ex. 1]. It remains to see that $\mathrm{Spec}(B) \to \mathrm{Spec}(A/\ker(f))$ is injective. If $p_1, p_2 \subseteq B$ are prime ideals with the same contraction to $A/\ker(f)$, and $x \in p_1$ is given, then for some $n \geq 0$ we have

$$x^n \in p_1 \cap (A/\ker(f)) \subseteq p_2,$$

hence $x \in p_2$ and $p_1 \subseteq p_2$. By symmetry this gives $p_1 = p_2$, as desired.

☐

Proposition 3.26. Suppose that $f\colon A \to B$ is a map of commutative rings such that:

(1) $f \otimes \mathbb{Q}$ is an isomorphism and
(2) for every prime p, $f \otimes \mathbb{Z}_{(p)}$ is an $\mathcal{F}_p$-isomorphism.

Then

$$f^*\colon \mathrm{Ring}(B, -) \to \mathrm{Ring}(A, -)$$

is an isomorphism after restriction to algebraically closed fields. In other words, f is a $\mathcal{V}$-isomorphism [GS99, Defn. A.3].

Proof. The first condition implies that f^* is an isomorphism after restricting to fields of characteristic 0. For algebraically closed fields of characteristic p, since $f \otimes \mathbb{Z}_{(p)}$ is an $\mathcal{F}_p$-isomorphism by assumption and any $\mathcal{F}_p$-isomorphism between two $\mathbb{F}_p$-algebras is a $\mathcal{V}$-isomorphism by [Qui71, Prop. B.8], we just need to verify that reducing a $\mathcal{F}_p$-isomorphism mod p induces a $\mathcal{F}_p$-isomorphism.

In other words, we need to show that if $f' = f \otimes \mathbb{Z}_{(p)}$ is an $\mathcal{F}_p$-isomorphism then so is $\overline{f} = f \otimes_{\mathbb{Z}} \mathbb{F}_p$. Suppose that $\overline{x} \in \ker(\overline{f})$, which we lift to $x \in A_{(p)}$. Now $f'(x) = pz$ for some $z \in B_{(p)}$. Since f' is an $\mathcal{F}_p$-isomorphism, there exists an $m \geq 0$ and $y \in A_{(p)}$ such that $z^{p^m} = f'(y)$. Now set $w = (x^{p^m} - p^{p^m} y)$ so

$$f'(w) = p^{p^m} z^{p^m} - p^{p^m} z^{p^m} = 0.$$

Since f' is an $\mathcal{F}_p$-isomorphism, w is nilpotent; however, w reduces to $\overline{x}^{p^m}$ so $\overline{x}$ is nilpotent.

Now consider some $\overline{z}' \in B \otimes \mathbb{F}_p$ and choose a lift $z' \in B_{(p)}$. Since f' is an $\mathcal{F}_p$-isomorphism there is a non-negative integer m' and $y' \in A_{(p)}$ such that $f'(y') = (z')^{p^{m'}}$. Reducing y' mod p, we see that $\overline{f}$ is an $\mathcal{F}_p$-isomorphism.

□

We will now combine the above results with the work of Quillen to obtain the following stratification result:

Proposition 3.27. Suppose that R is a homotopy commutative $\mathscr{F}$-nilpotent ring spectrum. Suppose further that $\pi_0^G R$ is noetherian and for every $H \in \mathscr{F}$, $\pi_0^H R$ is finite over $\pi_0^G R$ via Res_H^G. Then the canonical natural transformations

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} \mathrm{Ring}(\pi_0^H R, -) \to \mathrm{Ring}(\lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} \pi_0^H R, -) \xleftarrow{\mathrm{Res}_{\mathscr{F}}^{G\,*}} \mathrm{Ring}(\pi_0^G R, -)$$

are isomorphisms after restricting to algebraically closed fields. Similarly, the canonical maps between Zariski spaces

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} \mathrm{Spec}(\pi_0^H R) \to \mathrm{Spec}(\lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} \pi_0^H R) \xleftarrow{\mathrm{Res}_{\mathscr{F}}^{G\,*}} \mathrm{Spec}(\pi_0^G R)$$

are homeomorphisms.

Proof. The natural transformations directed to the right are isomorphisms by [Qui71, Lem. 8.11]. The natural transformations directed to the left are isomorphisms by Proposition 3.26 and Proposition 3.25, respectively. □

Example 3.28. Suppose $n(\mathscr{F}) = 1$. For instance, this occurs if G is a perfect group and $\mathscr{F} = \mathscr{P}$ is the family of all proper subgroups of G (Example 3.17).

In this case, the idempotent $e_{\mathscr{F}}$ belongs to the Burnside ring $A(G)$. We obtain a decomposition of the symmetric monoidal ∞-category Sp_G as

$$\mathrm{Sp}_G \simeq \mathcal{C}_1 \times \mathcal{C}_2,$$

where $\mathcal{C}_1$ consists of those G-spectra on which $e_{\mathscr{F}}$ is the identity (equivalently, is a self-equivalence), and $\mathcal{C}_2$ consists of those G-spectra on which $e_{\mathscr{F}}$ is null (cf. [Bar09]).

We claim that $\mathcal{C}_1$ is equal to the subcategories of $\mathscr{F}$-nilpotent, $\mathscr{F}$-complete, and $\mathscr{F}$-torsion G-spectra (which therefore all coincide). In particular, $\mathscr{F}$-nilpotence is a purely algebraic condition on the homotopy groups of a G-spectrum in this case.

We start by showing that every $\mathscr{F}$-complete G-spectrum belongs to $\mathscr{F}^{\mathrm{Nil}}$. In fact, this follows from Theorem 2.25, since our assumptions imply that the associated $\mathscr{F}$-spectral sequence has a horizontal vanishing line at E_2. We now invoke [MNN, Prop 4.21] to obtain that the subcategories of $\mathscr{F}$-nilpotent, $\mathscr{F}$-complete, and $\mathscr{F}$-torsion objects in Sp_G all coincide and that there is a splitting (of symmetric monoidal ∞-categories) of $\mathrm{Sp}_G \simeq \mathcal{C}_1' \times \mathcal{C}_2'$ where $\mathcal{C}_1'$ consists of the $\mathscr{F}$-nilpotent objects and $\mathcal{C}_2'$ consists of the $\mathscr{F}^{-1}$-local objects.

It remains to show that the two splittings of Sp_G coincide. To see this, observe that $e_{\mathscr{F}}$ restricts to 1 in $A(H)$ for $H \in \mathscr{F}$. As a result, $e_{\mathscr{F}}$ acts as the identity on $\{G/H_+\}_{H\in\mathscr{F}}$ and therefore on the localizing subcategory they generate. It follows that $\mathcal{C}_1$ contains the $\mathscr{F}$-torsion G-spectra, i.e., $\mathcal{C}_1 \supset \mathcal{C}_1'$. Conversely, if $X \in \mathrm{Sp}_G$ is $\mathscr{F}^{-1}$-local, then its restriction to Sp_H for $H \in \mathscr{F}$ is contractible; therefore the class $e_{\mathscr{F}} \in A(G)$, as a sum of classes induced from subgroups in $\mathscr{F}$, acts by zero. Therefore, $\mathcal{C}_2 \supset \mathcal{C}_2'$. It now follows that $\mathcal{C}_1 = \mathcal{C}_1', \mathcal{C}_2 = \mathcal{C}_2'$ as desired.

4. Defect bases and $\mathscr{F}$-split spectra

4.1. Classical defect bases and $\mathscr{F}$-split spectra.

Classical induction theory centers around the notion of a defect base. To define this, we will first need the following:

Proposition 4.1. Let $R(-)$ be a Green functor for the group G, see [Gre71, Web00]. Then there is a unique minimal family $\mathscr{F}$ such that the map

$$\mathrm{Ind}_{\mathscr{F}}^G \colon \bigoplus_{H\in\mathscr{F}} R(H) \to R(G) \tag{4.2}$$

is surjective.

Proof. It suffices to show that if $\mathscr{F}_1, \mathscr{F}_2$ are families such that $\mathrm{Ind}_{\mathscr{F}_1}^G$ and $\mathrm{Ind}_{\mathscr{F}_2}^G$ are surjective, then the same holds for $\mathscr{F}_1 \cap \mathscr{F}_2$. This is a straightforward exercise with the double coset formula [Gre71] which we leave to the reader. ☐

Definition 4.3. Let $R(-)$ be a Green functor for the group G. The *defect base of* R is the minimal family $\mathscr{F}$ of subgroups of G such that the map $\mathrm{Ind}_{\mathscr{F}}^G$ above (4.2) is surjective. The *defect base* of a G-ring spectrum R is the defect base of the Green functor $\pi_0^{(-)}R$.

To relate the notion of a defect base of a G-ring spectrum R, which only depends on R through $\pi_0^{(-)}R$, to the derived defect base, we have the following:

Proposition 4.4. Let R be a G-ring spectrum. For a family of subgroups $\mathscr{F}$ of G, consider the sum of the induction maps $\mathrm{Ind}_{\mathscr{F}}^G \colon \bigoplus_{H\in\mathscr{F}} \pi_0^H R \to \pi_0^G R$. Then the following are equivalent:

1. The map $\mathrm{Ind}_{\mathscr{F}}^G$ is surjective.
2. The product of the restriction maps
$$\prod_{H\in\mathscr{F}} \mathrm{Res}_H^G \colon R \to \prod_{H\in\mathscr{F}} F(G/H_+, R)$$
splits in Sp_G.
3. The G-spectrum R is $\mathscr{F}$-nilpotent and $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)}R) = 0$.
4. The G-spectrum R is $\mathscr{F}$-nilpotent and $\exp_{\mathscr{F}}(R) \leq 1$.

Proof. First we will prove (1) $\implies$ (4). By assumption, there is for each $H \in \mathscr{F}$ an element $m_H \in \pi_0^H R$ such that

$$1 = \sum_{H\in\mathscr{F}} \mathrm{Ind}_H^G m_H \in \pi_0^G R. \tag{4.5}$$

The element $m_H \in \pi_0^H R$ is represented by a G-map $R \to F(G/H_+, R)$ and $\mathrm{Ind}_H^G m_H$ is obtained by postcomposing with the projection $F(G/H_+, R) \simeq R \wedge G/H_+ \to R$. Assembling these together, we find that the composite map

$$R \xrightarrow{\prod_H m_H} \prod_{H \in \mathscr{F}} F(G/H_+, R) \to R$$

is homotopic to the identity. This retraction implies that R is $\mathscr{F}$-nilpotent with $\mathscr{F}$-exponent ≤ 1, proving (4).

The equivalence of (2) and (4) follows from Proposition 2.26 because $\prod_{H\in\mathscr{F}} F(G/H_+, R) \simeq F(\mathrm{sk}_0 E\mathscr{F}_+, R)$ with our preferred model for $E\mathscr{F}$.

Now we will show (2) $\implies$ (3). By assumption, R is a retract of the product spectrum $\prod_{H\in\mathscr{F}} F(G/H_+, R)$, which is $\mathscr{F}$-nilpotent. Since $\mathscr{F}^{\mathrm{Nil}}$ is closed under finite products and retracts we see that R is $\mathscr{F}$-nilpotent.

We will now show $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)} R) = 0$. Since the Amitsur-Dress-Tate cohomology groups are bigraded modules over the algebra in bidegree $(0,0)$, it suffices to prove this claim when the bidegree is $(0,0)$. Furthermore, since R is a retract of $\prod_{H\in\mathscr{F}} F(G/H_+, R)$, it suffices to show that for each $H \in \mathscr{F}$, $\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)} F(G/H_+, R)) = 0$, and by naturality again it suffices to consider the case $R = S^0$. However, in $\pi_0^G F(G/H_+, S^0)$, the unit is a norm from $\pi_0^H F(G/H_+, S^0)$, so that it vanishes in the $\mathscr{F}$-Tate cohomology, which is therefore zero.

Next, we prove (3) $\implies$ (1). Since R is $\mathscr{F}$-nilpotent, the $\mathscr{F}$-homotopy limit spectral sequence converges to $\pi_*^G R$. The vanishing of the Amitsur-Dress-Tate cohomology groups implies that this spectral sequence collapses at E_2 and the edge map induces an isomorphism $\pi_*^G R \cong \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi_*^H R$. Combining this with the identification of zeroth cohomology group from (3.11) we obtain

$$\widehat{H}^0_{\mathscr{F}}(\pi_0^{(-)} R) \cong \left(\lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} \pi_0^H R \right) / (\mathrm{Im}\,\mathrm{Ind}_{\mathscr{F}}^G) \cong \pi_0^G R / (\mathrm{Im}\,\mathrm{Ind}_{\mathscr{F}}^G).$$

Since these groups are zero by assumption, $\mathrm{Ind}_{\mathscr{F}}^G$ is surjective. □

Definition 4.6. We will say that a G-ring spectrum R is *$\mathscr{F}$-split* if it satisfies any of the equivalent characterizations of Proposition 4.4. More generally, we will say a G-spectrum M is *$\mathscr{F}$-split* if its endomorphism ring $\mathrm{End}(M)$ is $\mathscr{F}$-split.

Remark 4.7. It follows from the definitions that the defect base of a G-ring spectrum R is the smallest family $\mathscr{F}$ such that R is $\mathscr{F}$-split.

Remark 4.8. The $\mathscr{F}$-split condition can be used to test for projectivity and flatness (cf., [HKR00, Rem. 3.5.2]). For example, since KU is $\mathscr{F}$-split for the family $\mathscr{F}$ of Brauer elementary subgroups we know that for a G-spectrum X, $KU_*^G(X)$ is torsion-free if and only if $KU_*^H(X)$ is torsion-free for each $H \in \mathscr{F}$.

Proposition 4.9. Suppose that R is a G-ring spectrum such that $\pi_0^{(-)} R$ is isomorphic to the complex representation ring functor. Then the defect base of R is the family $\mathscr{F}$ of Brauer elementary subgroups of G, i.e., subgroups which are products of p-groups with cyclic subgroups of order prime to p for some prime p.

Proof. This purely algebraic claim about the representation ring Green functor is a combination of Brauer's theorem and its converse due to J. Green [Ser77, §11.3]. □

We now give two important cases in which the derived defect base and the defect base automatically coincide. Recall that a G-spectrum M is *connective* if, for every subgroup H of G, $\pi_i^H M = 0$ if $i < 0$.

Proposition 4.10. Suppose that R is a connective G-ring spectrum and $\mathscr{F}$ is a family of subgroups of G. Then R is $\mathscr{F}$-nilpotent if and only if R is $\mathscr{F}$-split. In particular, the defect base and the derived defect base of R coincide.

Proof. Clearly any $\mathscr{F}$-split spectrum is $\mathscr{F}$-nilpotent. For the other direction, suppose that R is $\mathscr{F}$-nilpotent, so the $\mathscr{F}$-homotopy colimit spectral sequence abuts to $\pi_*^G E\mathscr{F}_+ \wedge R \cong \pi_*^G R$ by Proposition 2.8. The connectivity assumption implies that $E^2_{0,0}$ is the only term contributing to $\pi_0^G R$ in this spectral sequence. Hence we obtain an isomorphism

$$\pi_0^G R \cong E^2_{0,0} \cong H_0^G(E\mathscr{F}_+; \pi_0^{(-)} R) \cong \operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} \pi_0^H R.$$

Since the E_2-edge map is an isomorphism on π_0^G, the E_1-edge map $\mathrm{Ind}_{\mathscr{F}}^G$ is surjective and hence R is $\mathscr{F}$-split. □

Proposition 4.11. Let R be a G-ring spectrum. If $n(\mathscr{F}) \in \pi_0^G R$ is a unit, then R is $\mathscr{F}$-nilpotent if and only if R is $\mathscr{F}$-split. In particular, the defect base and the derived defect base of R coincide.

Proof. If R is $\mathscr{F}$-split, then it is $\mathscr{F}$-nilpotent by definition. On the other hand, if R is $\mathscr{F}$-nilpotent then, by Proposition 4.4.(3), R is $\mathscr{F}$-split if and only if the Amitsur-Dress-Tate cohomology groups $\widehat{H}^*_{\mathscr{F}}(\pi_*^{(-)} R)$ vanish. Now since $n(\mathscr{F}) \in \pi_0^G R$ acts on these groups simultaneously by a unit and by zero by Proposition 3.12, we see that they must be zero. □

4.2. **Brauer induction theorems.** If we know the defect base of a G-ring spectrum we obtain an upper bound on the *derived* defect base. We now include results that enable us to go the other direction: if we know the derived defect base we obtain an upper bound on the defect base.

Proposition 4.12. Suppose that R is an $\mathscr{F}$-nilpotent G-ring spectrum. Let $\overline{\mathscr{F}} \supset \mathscr{F}$ be a family of subgroups satisfying the following condition: if $H \leq G$ that fits into a short exact sequence

$$e \to N \to H \to H/N \to e \tag{4.13}$$

where $N \in \mathscr{F}$ and H/N is a p-group, for some prime p, then $H \in \overline{\mathscr{F}}$. Then R is $\overline{\mathscr{F}}$-split.

Proof. Since $\mathscr{F} \subseteq \overline{\mathscr{F}}$ and R is $\mathscr{F}$-nilpotent, R is also $\overline{\mathscr{F}}$-nilpotent. By [Dre75, Prop. 1.6] we can find an $x \in \mathrm{Im}\,\mathrm{Ind}_{\overline{\mathscr{F}}}^G$ and $y \in \ker \mathrm{Res}_{\overline{\mathscr{F}}}^G$ such than $x + y = 1 \in \pi_0^G R$. Since y is nilpotent by Theorem 3.22, x must be a unit and $\mathrm{Ind}_{\overline{\mathscr{F}}}^G$ must be surjective as desired. □

Remark 4.14. Observe that any family $\overline{\mathscr{F}}$ satisfying the assumption of Proposition 4.12 necessarily contains all p-Sylow subgroups of G. So when R is a Borel-equivariant theory, the bound on the defect base of R given by Proposition 4.12 will provide no information (cf. Theorem 4.25 below).

When all of the subgroups in our given family $\mathscr{F}$ are *abelian*, and they often are, we can more explicitly identify a family $\overline{\mathscr{F}}$ satisfying the assumption of Proposition 4.12.

Proposition 4.15. Let $\mathscr{F} \subset \mathscr{A}$ be a family of abelian subgroups of G. Then the set $\overline{\mathscr{F}}$ of subgroups of the form $G' = H' \rtimes P$, where $P \in \underline{\mathscr{A}\ell\ell}$ is a p-group for some prime p and $H' \in \mathscr{F}[p^{-1}]$ is a subgroup in $\mathscr{F}$ of order prime to p, is a family of subgroups satisfying the assumption of Proposition 4.12.

Proof. Consider the family $\overline{\mathscr{F}'}$ of subgroups $H \leq G$ that fit into a short exact sequence as in (4.13) with $N \in \mathscr{F}$ and H/N a p-group (for some p). We will show that any such H belongs to $\overline{\mathscr{F}}$, so that $\overline{\mathscr{F}} = \overline{\mathscr{F}'}$. By Proposition 4.12, this will suffice for the result.

Note first that $N = N_1 \times N_2$ where N_2 is a p-group and $p \nmid |N_1|$, since N is abelian by assumption. Now $N_1 \le N$ is a characteristic subgroup and is therefore normal in H. Therefore, we obtain a new short exact sequence

$$e \to N_1 \to H \to H/N_1 \to e,$$

where $N_1 \in \mathscr{F}$ has order prime to p and H/N_1 is a p-group. The Schur-Zassenhaus theorem now implies that this extension splits. Consequently, $H \in \overline{\mathscr{F}}$ as desired. □

4.3. **Applications of $\mathscr{F}$-split spectra.** If a G-ring spectrum R is $\mathscr{F}$-split and M is an R-module, then the Amitsur-Dress-Tate cohomology groups of M vanish by Proposition 4.4. So under these hypotheses the $\mathscr{F}$-homotopy limit and colimit spectral sequences collapse at E_2 onto the zero line and we obtain the following integral form of Theorem 3.19:

Theorem 4.16. Let R be an $\mathscr{F}$-split G-ring spectrum and let X be a G-spectrum. Then for each R-module M, each of the following maps

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} M_H^*(X) \xrightarrow{\mathrm{Ind}_{\mathscr{F}}^G} M_G^*(X) \xrightarrow{\mathrm{Res}_{\mathscr{F}}^G} \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M_H^*(X) \tag{4.17}$$

$$\operatorname*{colim}_{\mathscr{O}(G)_{\mathscr{F}}} M_*^H(X) \xrightarrow{\mathrm{Ind}_{\mathscr{F}}^G} M_*^G(X) \xrightarrow{\mathrm{Res}_{\mathscr{F}}^G} \lim_{\mathscr{O}(G)_{\mathscr{F}}^{\mathrm{op}}} M_*^H(X) \tag{4.18}$$

is an isomorphism.

We will now show how the $\mathscr{F}$-split condition fits into Balmer's theory of descent for triangulated categories with a monoidal product which is exact in each variable [Bal12]. For this we suppose that R is an E_2-G-ring spectrum so the ∞-category $\mathrm{Mod}(R)$ of *structured* R-modules is monoidal [Lur14, Cor. 5.1.2.6]. Moreover, the monoidal product commutes with homotopy colimits in each variable; in particular, tensoring with a fixed module is an exact functor. It follows that the homotopy category $\mathrm{Ho}\,(\mathrm{Mod}(R))$ of R-modules is an idempotent-complete triangulated category with a monoidal structure that is exact in each variable (cf. [Man12]).

Now given a family of subgroups $\mathscr{F}$, let $X = \coprod_{H \in \mathscr{F}} G/H$ and $A = F(X_+, R)$. The R-algebra structure on A defines a monad T on $\mathrm{Ho}\,(\mathrm{Mod}(R))$ where $TM = A \wedge_R M$. The forgetful functor U_T from T-algebras in $\mathrm{Ho}\,(\mathrm{Mod}_R)$ to the underlying category $\mathrm{Ho}\,(\mathrm{Mod}_R)$ admits a right adjoint F_T. This defines a comonad $C = F_T U_T$ on the category of T-algebras and we let $\mathrm{Desc}_R(\mathscr{F})$ denote the category of C-coalgebras in T-algebras. The free algebra functor F_T canonically lifts to a functor

$$Q_{\mathscr{F}} : \mathrm{Ho}\,(\mathrm{Mod}(R)) \to \mathrm{Desc}_R(\mathscr{F})$$

and we will say that R *effectively descends along* $\mathscr{F}$ if $Q_{\mathscr{F}}$ is an equivalence of categories.

Proposition 4.19. Let R be an E_2-G-ring spectrum and $\mathscr{F}$ a family of subgroups. Then the following are equivalent:

(1) R effectively descends along $\mathscr{F}$.
(2) R is $\mathscr{F}$-split.

Proof. According to [Bal12, Cor. 3.1], R effectively descends along $\mathscr{F}$ if and only if $A \wedge_R (-)$ is faithful. Moreover, this condition is equivalent to the unit $R \to A$ admitting a retraction in $\mathrm{Ho}\,(\mathrm{Mod}(R))$. Indeed, if we have a retraction then clearly the functor is faithful; the other implication is [Bal12, Prop. 2.12] but we also include a more direct argument for the special case at hand: To see that $R \to A$ admits a retraction, we need to argue that the map $\mathrm{hofib}(R \to A) \to R$ is zero. We can check this after smashing with A, and hence it suffices to see that $A \cong R \wedge A \to A \wedge A$ admits a retraction; such is furnished by the multiplication map.

Finally, since the unit map

$$R \to A \simeq \prod_{H \in \mathscr{F}} F(G/H_+, R)$$

is the product of the restriction maps, we see that R is $\mathscr{F}$-split if and only if R effectively descends along $\mathscr{F}$. □

Example 4.20. In Proposition 4.9 we saw that KU is split for the family of Brauer elementary subgroups. When $G = \Sigma_3$, then the family of Brauer elementary subgroups is the family $\mathscr{C}$ of cyclic subgroups and the category $\mathscr{O}(\Sigma_3)_{\mathscr{C}}$ has a very simple form. So, in this case, one can more explicitly identify the target of the restriction isomorphism in Theorem 4.16.

For this purpose we fix Sylow subgroups C_2 and C_3 of Σ_3 and let $W_{\Sigma_3} C_3 \cong \mathbb{Z}/2$ denote the Weyl group of the 3-Sylow subgroup. We leave it to the reader to show (e.g., using Proposition A.6) that for any Σ_3-spectrum X, $KU^*_{\Sigma_3}(X)$ fits into the following pullback diagram:

(4.21)
$$\begin{array}{ccc} KU^*_{\Sigma_3}(X) & \xrightarrow{\mathrm{Res}^{\Sigma_3}_{C_3}} & KU^*_{C_3}(X)^{\mathbb{Z}/2} \\ {\scriptstyle \mathrm{Res}^{\Sigma_3}_{C_2}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^{C_3}_{e}} \\ W^*_{C_2}(X) & \xrightarrow[\mathrm{Res}^{C_2}_{e}]{} & KU^*_e(X)^{\Sigma_3} \end{array}$$

Here

$$W^*_{C_2}(X) := KU^*_{C_2}(X) \cap \left(\mathrm{Res}^{C_2}_e\right)^{-1} \left(KU^*_e(X)^{\Sigma_3}\right).$$

4.4. **Examples of $\mathscr{F}$-split spectra; the Borel-equivariant sphere.** We will now establish some more of the claims made in Figure 1.5B.

Proposition 4.22. Let T be a multiplicatively closed subset of $\mathbb{Z} \setminus \{0\}$. Then the derived defect base of $S[T^{-1}]$ is equal to its defect base, which is $\mathscr{A}\ell\ell$, the family of all subgroups.

Proof. Since $S[T^{-1}]$ is connective, the derived defect base of $S[T^{-1}]$ is the same as the defect base of $\pi_0^{(-)} S[T^{-1}] \cong A(-)[T^{-1}]$ (Proposition 4.10). It suffices to show that the family $\mathscr{P}$ of proper subgroups is not a defect base. The image of $\mathrm{Ind}^G_{\mathscr{P}}$ is generated as a $\mathbb{Z}[T^{-1}]$-module by those finite G-sets whose isotropy is a proper subgroup of G. In particular, they have trivial G-fixed points. It follows that $[G/G] \in A(G) \otimes \mathbb{Z}[T^{-1}]$ is not in the image of $\mathrm{Ind}^G_{\mathscr{P}}$ and hence $\mathscr{P}$ is not a defect base. □

Recall from Figure 1.5A that for a family $\mathscr{F}$, the subfamily $\underline{\mathscr{F}} \subseteq \mathscr{F}$ is defined to be the subset of groups in $\mathscr{F}$ of prime power order. Let $\mathscr{F}_{(0)} = \mathscr{T}$ and, for each prime p, let $\mathscr{F}_{(p)} \subseteq \mathscr{F}$ denote the subfamily of $\mathscr{F}$ whose elements are p-groups. For any multiplicative subset T of $\mathbb{Z} \setminus \{0\}$, we let $\underline{\mathscr{F}}[T^{-1}]$ denote the subset of groups in $\underline{\mathscr{F}}$ whose order is not invertible in $\mathbb{Z}[T^{-1}]$.

Definition 4.23. The *constant* Green functor $R(-)$ associated to a ring R is defined as follows:

- For each subgroup H of G, $R(G/H)$ is the ring R.
- The restriction and conjugation maps of $R(-)$ are all identities.
- For each chain of subgroup inclusions $H < K < G$, Ind^K_H is multiplication by $|K/H|$.

Proposition 4.24. Let $R \neq 0$ be a ring and let T be the set of elements in $\mathbb{Z}$ which are invertible in R. Then $\underline{\mathscr{A}\ell\ell}[T^{-1}]$ is the defect base of both HR and $\underline{HR}$; $\underline{\mathscr{A}\ell\ell}[T^{-1}]$ is also the derived defect base of HR.

Proof. Since HR is connective, its derived defect base and its defect base are equal (Proposition 4.10). The remaining claims follow from since $\pi_0^{(-)} HR \cong \pi_0^{(-)} \underline{HR} = R$ is the constant Green functor at R. Indeed, the image of $\mathrm{Ind}_{\mathscr{F}}^G$ inside $R(G/G) = R$ is the principal ideal generated by $\gcd(\{|G/H|\}_{H\in\mathscr{F}})$, so $\mathrm{Ind}_{\mathscr{F}}^G$ is surjective if and only if this integer is a unit in R. □

We now give a particularly deep example of the determination of the derived defect base of a G-ring spectrum. We do not know if it is possible to carry this out without the use of the Segal conjecture.

Theorem 4.25. The derived defect base of $\underline{S}$ is equal to its defect base $\mathscr{A\ell\ell}$. More generally if $T \subseteq \mathbb{Z} \setminus \{0\}$ is a multiplicatively closed subset, then the derived defect base of $\underline{S[T^{-1}]}$ is equal to its defect base $\mathscr{A\ell\ell}[T^{-1}]$. Consequently, if M is an $\underline{S[T^{-1}]}$-module, then $\underline{M}$ is $\mathscr{A\ell\ell}[T^{-1}]$-nilpotent.

Proof. We first bound *above* the derived defect base of $\underline{S[T^{-1}]}$ as claimed. In fact, it suffices (cf. [MNN, Prop. 6.40]) to show that:

(1) $\underline{S}$ is nilpotent (even split) for the family $\mathscr{A\ell\ell}$.
(2) If G is a p-group for a prime number p invertible in $\mathbb{Z}[T^{-1}]$, then $\underline{S[T^{-1}]}$ is nilpotent (even split) for the family consisting of the trivial group.

Recall first that the category of dualizable structured modules over $\underline{S[T^{-1}]}$ embeds into $\mathrm{Fun}(BG, \mathrm{Perf}(S[T^{-1}]))$ (where the latter $S[T^{-1}]$ belongs to the category of non-equivariant spectra) by [MNN, Cor. 6.21].

We now treat the assertions above. We begin with the first. For each prime p dividing the order of the group G, we let $G_p \leq G$ denote a p-Sylow subgroup. We consider the composite map

$$\psi_p \colon \underline{S} \xrightarrow{\mathrm{Ind}_{G_p}^G} \underline{(G/G_p)_+} \xrightarrow{\mathrm{Res}_{G_p}^G} \underline{S},$$

which induces multiplication by $|G/G_p|$ on the underlying spectrum. The orders of the $\{|G/G_p|\}$ generate the unit ideal in $\mathbb{Z}$, so there is a linear combination $\sum_{p||G|} n_p \psi_p$ of the ψ_p which is a self-equivalence ϕ of $\underline{S}$. We thus get a retraction diagram

$$\underline{S} \xrightarrow{\Sigma n_p \mathrm{Ind}_{G_p}^G} \bigoplus_p \underline{(G/G_p)_+} \xrightarrow{\phi^{-1}\circ \Sigma \mathrm{Res}_{G_p}^G} \underline{S},$$

which shows that $\underline{S}$ is nilpotent (even split) for the family $\mathscr{A\ell\ell}$. For the second claim, we observe that in this case the composite

$$\underline{S[T^{-1}]} \xrightarrow{\mathrm{Ind}_e^G} \underline{G_+[T^{-1}]} \xrightarrow{\mathrm{Res}_e^G} \underline{S[T^{-1}]}$$

is an equivalence.

Finally, we show that the families are minimal as claimed. In general, if a prime p is not inverted in $\underline{S[T^{-1}]}$, then $\underline{S_{(p)}}$ is an $\underline{S[T^{-1}]}$-module. It thus suffices to show that if G is a p-group, then $\underline{S_{(p)}}$ is *not* nilpotent for the family of proper subgroups. In the next proposition, we will show that $\underline{S/p}$ is not nilpotent for the family of proper subgroups, which will prove the claim. Note also that the since the derived defect base of $\underline{S[T^{-1}]}$ is $\mathscr{A\ell\ell}[T^{-1}]$ and $\underline{S[T^{-1}]}$ is split for this family, that is also the defect base. □

Remark 4.26. The Segal conjecture determines $\pi_0^{(-)}$ of the Borel-equivariant sphere as a completion of the Burnside Green functor. Using general facts about the Burnside ring, one can check directly that if G is a p-group, then $\underline{S}$ is not nilpotent for the family of proper subgroups (in fact, $(\underline{S})_{\mathbb{Q}}$ is not). We leave the details to the reader.

Proposition 4.27. Let p be a prime and X a nontrivial finite p-torsion spectrum. Then the derived defect base of $\underline{X}$ is $\mathscr{A}\ell\ell_{(p)}$.

Proof. By assumption, $\underline{X}$ is an $\overline{S_{(p)}}$-module and hence $\mathscr{A}\ell\ell_{(p)}$-nilpotent, so we just need to prove the minimality of this family. For this claim, it suffices to show that if G is a p-group then $\Phi^G \underline{X} \not\simeq *$.

Let $i_* : \mathrm{Sp} \to \mathrm{Sp}_G$ denote the functor that sends the sphere to the G-sphere. The class of spectra Y such that the canonical map $i_* Y \to \underline{Y}$ is an equivalence of G-spectra forms a thick subcategory $\mathcal{C} \subseteq \mathrm{Sp}$. Since G is a p-group, the Segal conjecture [Car84] (in the equivalent form given in [MM82, Prop. B]) implies $S/p \in \mathcal{C}$ and hence every finite p-torsion spectrum belongs to $\mathcal{C}$. Since X is nontrivial and $\Phi^G i_* X \simeq X$, it follows that $\Phi^G \underline{X} \simeq \Phi^G i_* X \simeq X \not\simeq *$. □

5. Derived defect bases via orientations

In this final section, we give the main examples of derived defect bases. All of these will rely on the use of *orientations* together with thick subcategory arguments, and a reduction (following Quillen) to the family of *abelian* subgroups via consideration of the flag variety.

5.1. On G-spectra with Thom isomorphisms.

We will now consider how our conditions on a homotopy commutative G-ring spectrum simplify when the spectrum is oriented in the following sense (Cf. [GM97, Defn. 2.1, Rem. 2.2, Defn. 3.7]) :

Definition 5.1. Let R be a homotopy commutative G-ring spectrum and V an orthogonal representation of G. Then a *Thom class for V* with respect to R is a map of G-spectra $\mu_V \colon S^{V-|V|} \to R$ such that its canonical extension to an R-module map

$$R \wedge S^{V-|V|} \xrightarrow{R\wedge \mu_V} R \wedge R \xrightarrow{\mu} R$$

is an equivalence. If V is a representation of a subgroup $H \leq G$, then we say that a *Thom class for V* with respect to R is a Thom class for V with respect to the H-spectrum $\mathrm{Res}^G_H R$.

Let $\mathcal{I}$ be a class of representations closed under finite direct sums, restriction, and conjugation (e.g., unitary, oriented, $8n$-dimensional spin). We will say that R has *multiplicative Thom classes* for $\mathcal{I}$ if, for each subgroup $H \leq G$, it has Thom classes for every H-representation V in $\mathcal{I}$ and the Thom classes are multiplicative: $\mu_{V\oplus W} = \mu_V \cdot \mu_W$.

In this case we define the *oriented Euler class of* V, $\chi(V) \in R^{|V|}_H(*)$, to be the following composition

$$\chi(V)\colon S^{-|V|} \xrightarrow{e_V \wedge S^{-|V|}} S^{V-|V|} \xrightarrow{\mu_V} \mathrm{Res}^G_H R.$$

Remark 5.2. Thom classes often appear in another guise which we will now describe (cf. [GM97, Rem. 2.2]). Given an orthogonal H-vector bundle V on an H-space X we have an associated Thom space TV. Suppose that we are given a family of isomorphisms of R^*_H-modules

$$\phi_V \colon R^*_H(\Sigma^{|V|}(X_+)) \to R^*_H(TV)$$

which are natural in X and H. In the case X is a point, $TV \simeq S^V$ and we can rewrite ϕ_V as an isomorphism

$$\pi^H_* R \cong \pi^H_* R \wedge S^{|V|-V}$$

of $\pi^H_* R$-modules. The unit in $\pi^H_0 R$ corresponds to a map of H-spectra $\mu_V \colon S^{V-|V|} \to \mathrm{Res}^G_H R$ which extends to an equivalence of $\mathrm{Res}^G_H R$-modules

$$\mathrm{Res}^G_H R \wedge S^{V-|V|} \to \mathrm{Res}^G_H R.$$

So we see that natural Thom isomorphisms give rise to such Thom classes.

In terms of the $RO(G)$-graded groups $\pi_\star^G R$, the Euler classes are evidently related to the oriented Euler classes by the following formula:

$$\chi(V) = e_V \cdot \mu_V.$$

Since the Euler classes are in the Hurewicz image of $\pi_\star^G S$, they are necessarily central and it follows that $\chi(V^n) = \chi(V)^n$. Since μ_V is necessarily a unit in $\pi_\star^G R$, we immediately obtain the following:

Lemma 5.3. Suppose that R is a homotopy commutative G-ring spectrum with multiplicative Thom classes for a class $\mathcal{I}$ of representations and V admits the structure of an $\mathcal{I}$-representation of G. Then, for each positive integer n, $\chi(V)^n$ is zero if and only if e_V^n is zero.

The following proposition will play a key role in proving the $\mathscr{F}$-nilpotency of many G-spectra.

Proposition 5.4. Suppose that R is a homotopy commutative G-ring spectrum with multiplicative Thom classes for either unoriented, oriented, unitary, or $8n$-dimensional spin representations. Then the following are equivalent:

(1) The G-spectrum R is $\mathscr{F}$-nilpotent.
(2) For every subgroup $H \leq G$ with $H \notin \mathscr{F}$, if

$$x \in \ker \left(\pi_*^H R \to \prod_{K<H} \pi_*^K R \right),$$

then x is nilpotent.

Proof. The implication (1) $\implies$ (2) is an easy consequence of Theorem 3.22, since for $H \notin \mathscr{F}$, $\mathrm{Res}_H^G R$ is nilpotent for the family of proper subgroups of H.

For the reverse implication, it suffices by Corollary 2.5 to show that for each $H \notin \mathscr{F}$, $e_{\tilde{\rho}_H} \in \pi_\star^H R$ is nilpotent. This class restricts to zero on all proper subgroups of H by the first part of Proposition 2.2, so if assumption (2) were stated in terms of $RO(H)$-graded groups we would already be done. Instead, we will use the Thom isomorphisms to reduce to the integer grading: By Lemma 5.5 below there is an n such that $n\tilde{\rho}_H$ has an Euler class $\chi(n\tilde{\rho}_H) \in \pi_*^H R$. This class is nilpotent by Lemma 5.3 and assumption (2). Since the nilpotency of $e_{\tilde{\rho}_H}$ is equivalent to the nilpotency of $\chi(n\tilde{\rho}_H)$ by Lemma 5.3, the result follows. □

Lemma 5.5. For any finite group G,

(1) $2\tilde{\rho}_G$ underlies a unitary and hence oriented representation and
(2) $8\tilde{\rho}_G$ underlies a spin representation whose dimension is divisible by 8.

Proof. The real representation underlying the complex reduced regular representation of G is $2\tilde{\rho}_G$, which proves the first claim.

Now $8\tilde{\rho}_G$ admits a spin structure if and only if the first two Stiefel-Whitney classes

$$w_1(8\tilde{\rho}_G), w_2(8\tilde{\rho}_G) \in H^*(BG; \mathbb{F}_2)$$

vanish. By the Whitney sum formula $w_1(8\tilde{\rho}_G) = 2w_1(4\tilde{\rho}_G) = 0$ and hence $w_2(8\tilde{\rho}_G) = 2w_2(4\tilde{\rho}_G) = 0$ as desired.

□

5.2. **Equivariant topological K-theory.**

Proposition 5.6. The derived defect base of both the complex and real equivariant K-theory spectra, KU and KO, is $\mathscr{C}$.

Proof. First we show that KO and KU are $\mathscr{C}$-nilpotent. Since KU is a KO-module it suffices to prove this for KO. Now KO admits multiplicative Thom classes for $8n$-dimensional spin representations and is 8-periodic [Ati68, Thm. 6.1], so by Proposition 5.4, it suffices to show that any element $x \in \pi_0^H KO = RO(H)$ which restricts to zero on all cyclic subgroups is zero. By elementary character theory, the complexification of x in $R(H)$ is zero since it is zero on all cyclic subgroups. Now since the composite $RO(H) \to R(H) \to RO(H)$ of the complexification and the forgetful map is multiplication by 2 and $RO(H)$ is torsion-free, x is necessarily zero.

To see that this is a minimal family it suffices to prove this for the KO-module KU. Now KU admits multiplicative Thom classes for unitary representations and is 2-periodic [Ati68, Thm. 4.3]. So it suffices, by Proposition 5.4 again, to construct, for an arbitrary cyclic group G, a non-nilpotent element $x \in R(G)$ which restricts to zero on all proper subgroups. Since

$$R(H_1 \times H_2) \cong R(H_1) \otimes R(H_2),$$

if we can do this in the case $G = C_{p^n}$ is a cyclic p-group, then we can tensor the classes together to obtain the desired element.

The character map $R(C_{p^n}) \to \prod_{g \in C_{p^n}} \mathbb{C}$ is a ring map and an injection into a reduced ring, so any non-zero element of $R(C_{p^n})$ is non-nilpotent. Now $R(C_{p^n}) = \mathbb{Z}[z_n]/(z_n^{p^n} - 1)$ and $x = z_n^{p^{n-1}} - 1$ is a nontrivial and hence non-nilpotent element of $R(C_{p^n})$. Under the inclusion $C_{p^{n-1}} \to C_{p^n}$ of the unique maximal proper subgroup, z_n restricts to $z_{n-1} \in R(C_{p^{n-1}})$ and we see that x restricts to zero on this subgroup. It thus restricts to zero on all proper subgroups as desired. □

Proposition 5.7. The derived defect base of both the Borel-equivariant K-theory spectra $\underline{KU}$ and $\underline{KO}$ is $\underline{\mathscr{C}}$.

Proof. First we will show that these spectra are $\underline{\mathscr{C}}$-nilpotent. Since the arguments for the real and complex case are identical, we will just do the complex case. Since KU is split [May96, §XVI.2][LMS86, p. 458], $\underline{KU} = F(EG_+, i_* KU) \simeq F(EG_+, KU)$. It follows that $\underline{KU}$ is a KU-module and hence $\mathscr{C}$-nilpotent by Proposition 5.6. Since $\underline{KU}$ is also an $\underline{S}$-module it is $\mathscr{C} \cap \mathscr{A\ell\ell} = \underline{\mathscr{C}}$-nilpotent by Theorem 4.25.

We will now prove that this is the minimal such family for $\underline{KU}$. Since $\underline{KU}$ is a $\underline{KO}$-module this will establish the minimality claim for $\underline{KO}$ as well. Now for each cyclic p-group G, we will construct a non-nilpotent element $x \in \pi_0^G \underline{KU}$, which restricts to zero on all proper subgroups. Since KU is complex-orientable, it has Thom classes for unitary representations and the minimality claim will then follow from Proposition 5.4.

We have already constructed such an x in $\pi_0^G KU \cong R(G)$ in Proposition 5.6, so it suffices to show that the natural ring map

$$i\colon \pi_0^G KU \to \pi_0^G \underline{KU}$$

is an injection. By the Atiyah-Segal completion theorem [AS69], $\pi_0^G \underline{KU}$ is $\widehat{R}(G)$, the completion of $R(G)$ at the ideal of virtual representations of dimension zero, and i is the completion map. This map is an injection for all p-groups G by [Ati61, Prop. 6.11], so the claim follows. □

Remark 5.8. One can also show that the derived defect bases of KU and KO (resp. $\underline{KU}$ and $\underline{KO}$) agree with an independent argument using Galois descent [MNN, Prop. 9.15].

There are at least two standard notions of 'connective' equivariant K-theory. The first is $KU_{\tau \geq 0}$ which is the standard connective cover: it admits a canonical map $KU_{\tau \geq 0} \to KU$ such that $\pi_i^{(-)}$ is an isomorphism when $i \geq 0$ and $\pi_i^{(-)} KU_{\tau \geq 0} = 0$ for $i < 0$. By Proposition 4.10 the derived defect base of $KU_{\tau \geq 0}$ is its defect base and this is the family of Brauer elementary subgroups of G by Proposition 4.9. The more interesting variant is the following:

Definition 5.9. Let ku denote the equivariant connective[6] K-theory spectrum constructed by Greenlees [Gre04, Gre05]. This is defined by the following homotopy pullback along the self-evident ring maps:

$$\begin{array}{ccc} ku & \longrightarrow & KU \\ \downarrow & & \downarrow \\ \underline{ku} & \longrightarrow & \underline{KU} \simeq F(EG_+, KU) \end{array} \tag{5.10}$$

The real analogue, ko is defined similarly.

Proposition 5.11. The derived defect base of ku and ko is $\mathscr{C} \cup \mathscr{E}$.

Proof. Since the arguments for ku and ko are essentially identical, we will prove the claim for ku. We have already shown in Propositions 5.6 and 5.7 that the derived defect bases of KU and $\underline{KU}$ are $\mathscr{C}$ and $\underline{\mathscr{C}}$ respectively. In Corollary 5.34 we will show that the derived defect base of $\underline{ku}$ is $\underline{\mathscr{C}} \cup \mathscr{E}$. It follows that each of these spectra is $\mathscr{C} \cup \mathscr{E}$-nilpotent. Since the ∞-category of $\mathscr{C} \cup \mathscr{E}$-nilpotent G-spectra is closed under homotopy pullbacks, ku is $\mathscr{C} \cup \mathscr{E}$-nilpotent. The required results for KO, $\underline{KO}$, and $\underline{ko}$ are proven in Propositions 5.6 and 5.38.

Since KU and $\underline{ku}$ are ku-algebras via the above maps, the minimality claim follows from the minimality of the families for $\underline{ku}$ and KU. □

Let G be a compact Lie group with an involution $g \mapsto \overline{g}$, i.e., a *Real Lie group* in Atiyah's terminology. Then one can form a split extension of groups (the semidirect product)

$$1 \to G \to \widetilde{G} \to C_2 \to 1.$$

A $\widetilde{G}$-space is then a *Real G-space* in the sense of [AS69, §6]. There is an equivariant cohomology theory $K\mathbb{R}^*_G$ on $\widetilde{G}$-spaces X such that, for a finite $\widetilde{G}$-CW complex, $K\mathbb{R}^0_G(X)$ is the Grothendieck group of *Real G-vector bundles on X*. In [HJJS08, Ch. 14], the Thom isomorphism theorem is proved for Real G-vector bundles on compact $\widetilde{G}$-spaces. We let $K\mathbb{R}_G$ be a ring $\widetilde{G}$-spectrum representing this cohomology theory.

We have the following generalization of Proposition 2.14.

Proposition 5.12. Suppose G (and therefore $\widetilde{G}$) is finite. The derived defect base of the $\widetilde{G}$-spectrum $K\mathbb{R}_G$ is given by the family of cyclic subgroups of G.

Proof. We will need the two following observations:

1. Let X be a finite $\widetilde{G}$-CW complex on which G acts trivially, so that it arises from a finite C_2-CW complex. Then we have $K\mathbb{R}^*_G(X) = K\mathbb{R}^*(X) \otimes_{\mathbb{Z}} K\mathbb{R}^0_G(*)$.
2. We have $\mathrm{Res}^{\widetilde{G}}_G K\mathbb{R}_G = KU_G = KU$.

By the second item, it suffices to show that $K\mathbb{R}_G$ is nilpotent for the family of subgroups of G. To show this, we first let σ denote the real sign representation of C_2, regarded as a $\widetilde{G}$-representation. Then the first item together with the calculation used in Proposition 2.14 shows that the Euler class $S^0 \to S^{3\sigma}$ becomes null-homotopic after smashing with $K\mathbb{R}_G$. This means that $K\mathbb{R}_G$ is a retract of $S(3\sigma)_+ \wedge K\mathbb{R}_G$ and is therefore nilpotent for the family of subgroups of G. □

Remark 5.13. We do not know of an extension of Proposition 2.15 along the lines of Proposition 5.12 as of this time.

[6] Which is not generally connective!

5.3. **Complex-oriented Borel-equivariant theories.** The following is fundamental to all of the following calculations of derived defect bases of Borel-equivariant spectra.

Theorem 5.14 (cf. [HKR00] and [MNN, Cor. 7.48])**.** *The derived defect base of the Borel-equivariant G-spectrum $\underline{MU}$ is $\mathscr{A}$.*

Proof. We begin by showing that $\underline{MU}$ is $\mathscr{A} = \mathscr{A}\ell\ell \cap \mathscr{A}$-nilpotent. Since $\underline{MU}$ is an $\underline{S}$-module and the latter is $\mathscr{A}\ell\ell$-nilpotent, it suffices to show that $\underline{MU}$ is $\mathscr{A}$-nilpotent. Since $\underline{MU}$ has Thom isomorphisms for unitary representations, it suffices by Proposition 5.4 to show that any x in $MU^*(BG)$ which restricts to zero on each abelian subgroup is nilpotent.

The following nilpotence argument is standard (see [GS99, §4]) and dates back to [Qui71]. Let F be the variety of complete flags associated to a faithful representation of G. This is a compact G-manifold with abelian isotropy, so it admits the structure of a finite G-CW complex, whose cells have abelian isotropy which we will now fix. By [HKR00, Prop. 2.6] we have an inclusion $MU^*(BG) \to MU^*(EG \times_G F)$. So it suffices to show that x is nilpotent in the target ring. Filtering F by its G-CW structure, there is a multiplicative spectral sequence

$$E_2^{s,t} = H^s_G(F; \pi_{-t}^{(-)} \underline{MU}) \Longrightarrow MU^{t+s}(EG \times_G F)$$

with the following properties:

(1) Any class $y \in MU^*(BG)$ which restricts to zero in $MU^*(BA)$ for each abelian subgroup A belongs to the kernel of the edge homomorphism

$$MU^*(EG \times_G F) \to E_2^{0,*} \subseteq E_1^{0,*}.$$

This is a consequence of the following two facts:

(1.1) The flag variety F has abelian isotropy and hence $E_1^{0,*}$ is a product of terms of the form $MU^*(EG \times_G G/A) \cong MU^*(BA)$.

(1.2) The E_1-edge homomorphism is the product of the restriction homomorphisms induced by a coproduct of projections of the form $G/A \to G/G$.

(2) $E_2^{s,*} = 0$ for $s > \dim F$ by definition of the spectral sequence.

Property (1) shows that x must be detected in positive filtration, while the property (2) shows that x is nilpotent.

In Proposition 5.36 below, we will show that for every prime p and integer n, the $\underline{MU}$-module $\underline{E_n}$ has $\mathscr{A}^n_{(p)}$ as its derived defect base. Since n and p are arbitrary this forces the minimality of the family for $\underline{MU}$. □

Remark 5.15. The argument with the flag variety above plays a key role in the unipotence results of [MNN, §7]. A consequence of those results (combined with Proposition 2.26 above) is that if G admits an n-dimensional faithful complex representation then we obtain an explicit upper bound

$$\exp_{\mathscr{A}}(\underline{MU}) \leq n(n-1)+1$$

on the $\mathscr{A}$-exponent of G-equivariant $\underline{MU}$.

5.4. **Ordinary Borel-equivariant cohomology.** We will now discuss the further reduction one can make when one is over the integers.

Proposition 5.16 (cf. [Qui71])**.** *The derived defect base of $\underline{H\mathbb{F}_p}$ is $\mathscr{E}_{(p)}$.*

Proof. We first prove that $\underline{H\mathbb{F}_p}$ is $\mathscr{E}_{(p)}$-nilpotent. Since $\underline{H\mathbb{F}_p}$ is an $\underline{MU}$-module and an $\underline{S_{(p)}}$-module, $\underline{H\mathbb{F}_p}$ is $\mathscr{A} \cap \mathscr{A}\ell\ell_{(p)} = \mathscr{A}_{(p)}$-nilpotent by Theorems 5.14 and 4.25. Moreover, $\underline{H\mathbb{F}_p}$ has Thom isomorphisms for oriented representations. So by Proposition 5.4, it suffices to show that if $G = A$ is an abelian p-group and $x \in H^*(BA; \mathbb{F}_p)$ restricts to 0 on each elementary abelian subgroup then x is nilpotent.

The remainder of the argument follows from elementary group cohomology calculations: If $A = C_{p^{i_1}} \times \cdots \times C_{p^{i_n}}$, then there is a polynomial subalgebra $R = \mathbb{F}_p[x_1, \ldots, x_n] \subset H^*(BA; \mathbb{F}_p)$ whose generators are in degree 2 and such that for any element $x \in H^*(BA; \mathbb{F}_p)$, $x^p \in R$. Moreover, there is a maximal elementary abelian subgroup E of A such that the composite

$$R \to H^*(BA; \mathbb{F}_p) \xrightarrow{\mathrm{Res}^A_E} H^*(BE; \mathbb{F}_p)$$

is an injection. It follows that if x restricts to zero on E then x is nilpotent.

To prove minimality of the family $\mathscr{E}_{(p)}$, we suppose that $G = E$ is an elementary abelian group. To see that $\underline{H\mathbb{F}_p}$ is not $\mathscr{P}$-nilpotent we will construct an element $z \in H^*(BE; \mathbb{F}_p)$, which restricts to zero on each proper subgroup of E and belongs to the polynomial subalgebra R of $H^*(BE; \mathbb{F}_p)$ and hence is non-nilpotent. Let $y \in H^1(BC_p; \mathbb{F}_p) = \mathbb{F}_p$ be non-zero. For each nontrivial homomorphism $\phi \colon E \to C_p$, we obtain a nontrivial element $y_\phi = \beta\phi^*(y) \in R \cap H^2(BE; \mathbb{F}_p)$. By construction y_ϕ restricts to zero on the maximal proper subgroup $\ker \phi$ of E. Since any proper subgroup is contained in the kernel of such a map, the element

$$z = \prod_{\phi \in \mathrm{Gp}(E, C_p) \setminus \{0\}} y_\phi$$

restricts to zero on any proper subgroup of E and is non-nilpotent as desired because $z \in R$. □

Corollary 5.17. The derived defect base of $\underline{MO}$ is $\mathscr{E}_{(2)}$.

Proof. Since $H\mathbb{F}_2$ is an MO-algebra via the zeroth Postnikov section, the minimality claim will follow from the minimality claim for $\underline{H\mathbb{F}_2}$ in Proposition 5.16. Moreover, MO additively splits into a wedge of suspensions of $H\mathbb{F}_2$ and hence admits the structure of an $H\mathbb{F}_2$-module. It follows that $\underline{MO}$ is $\mathscr{E}_{(2)}$-nilpotent. □

Example 5.18. We now examine the $\mathscr{E}_{(2)}$-homotopy limit spectral sequence for $\underline{H\mathbb{F}_2}$ when $G = Q_8$ is the quaternion group of order 8. The edge homomorphism of this spectral sequence was first analyzed by Quillen [Qui71, Ex. 7.4] and provides an example where this map is neither an injection nor a surjection, but is evidently an $\mathcal{F}_2$-isomorphism. We will now calculate the rest of the spectral sequence and verify Quillen's result.

In this case, the only nontrivial elementary abelian subgroup is the center $Z(Q_8) = C_2$. Since this is normal with quotient $C_2 \times C_2$, by Lemma A.3 the $\mathscr{E}_{(2)}$-homotopy limit spectral sequence (which is also the Lyndon-Hochschild-Serre spectral sequence) takes the form

$$H^s(B(C_2 \times C_2); H^t(BC_2; \mathbb{F}_2)) \Longrightarrow H^{t+s}(BQ_8; \mathbb{F}_2).$$

Since the action of $C_2 \times C_2$ on the center is trivial, the local coefficient system is trivial.

Hence the E_2-page is isomorphic to $\mathbb{F}_2[e_1, e_2] \otimes \mathbb{F}_2[e_3]$ where e_3 generates the cohomology of the center and is in bidegree $(0, 1)$. Now e_1 and e_2 are both in bidegree $(1, 0)$ and for degree reasons they are permanent cycles. Since the spectral sequence does not have a horizontal vanishing line at the E_2-page we know that the last remaining indecomposable e_3, must support a differential. For degree reasons this must be a d_2.

To identify this differential we note that the $\mathscr{E}_{(2)}$-homotopy limit spectral sequence is acted on by $\mathrm{Aut}(Q_8)$. This follows from the observation that the family $\mathscr{E}_{(2)}$ of elementary abelian 2-groups is invariant under automorphisms of Q_8, and all resolutions in question can therefore be carried out respecting the $\mathrm{Aut}(Q_8)$-action. Since $\mathrm{Aut}(Q_8)$ fixes the center and acts transitively on the non-zero elements of the quotient group $C_2 \times C_2$ [AM04, Lem. IV.6.9] we see that $d_2(e_3)$ must land in the invariants

$$H^2(BC_2 \times C_2; \mathbb{F}_2)^{\mathrm{Aut}(Q_8)} \cong \mathbb{F}_2\{e_1^2 + e_1e_2 + e_2^2\}.$$

This forces $d_2(e_3) = e_1^2 + e_1e_2 + e_2^2$.

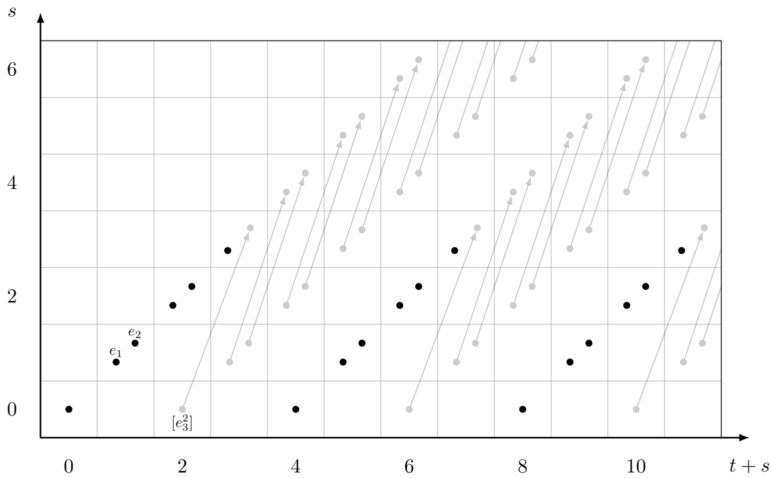

FIGURE 5.19. The E_3-page of the $\mathscr{E}_{(2)}$-homotopy limit spectral abutting to $H^{t+s}(BQ_8;\mathbb{F}_2)$.

The E_3-page shown in Figure 5.19 does not yet have a horizontal vanishing line so there must be a further differential. By the same reasoning as above we see that $[e_3^2]$ must support a differential and this must be a d_3 which lands in the invariants of the $\mathrm{Aut}(Q_8)$-action. This forces $d_3([e_3^2]) = e_1^2e_2 + e_1e_2^2$. At this point there is no room for further differentials and the spectral sequence collapses at E_4. There are no additive or multiplicative extensions for degree reasons. So we obtain:

$$H^*(BQ_8;\mathbb{F}_2) \cong \mathbb{F}_2[e_1, e_2, [e_3^4]]/(e_1^2 + e_1e_2 + e_2^2, e_1^2e_2 + e_1e_2^2)$$

(cf. [AM04, Lem. IV.2.10]). Since there are elements of filtration 3 at E_∞, we find that $\exp_{\mathscr{E}_{(2)}}(\underline{H\mathbb{F}_2}) \geq 4$.

We can in fact show that this is an equality, equivalently, that there is a 3-dimensional finite Q_8-CW-complex X with isotropy in $\mathscr{E}_{(2)}$ such that $\underline{H\mathbb{F}_2}$ splits off $\underline{H\mathbb{F}_2} \wedge X_+$. For this, we choose $X = \mathbb{P}(\mathbb{H})$, the projective space of the 4-dimensional real representation of Q_8 afforded by quaternion multiplication on $\mathbb{H} \cong \mathbb{R}^4$. The required splitting follows from the projective bundle theorem in mod 2-cohomology (cf. [Hus94, Sec. 17, Thm. 2.5]). In fact, this produces a map

$$\bigvee_{i=0}^{3} \Sigma^{-i}\underline{H\mathbb{F}_2} \to \underline{H\mathbb{F}_2} \wedge \mathbb{D}X_+, \tag{5.20}$$

classifying the generators of the free $H^*(BQ_8;\mathbb{F}_2)$-module $\pi_*^{Q_8}(\underline{H\mathbb{F}_2} \wedge \mathbb{D}X_+) \simeq H^*_{Q_8}(X;\mathbb{F}_2)$. Since the projective bundle formula implies that (5.20) is an equivalence on H-fixed points for any $H \leq Q_8$, we get that (5.20) is an equivalence and we have the desired splitting after dualizing.

Remark 5.21. In [Qui71], Quillen actually considers a smaller indexing category than $\mathscr{O}(G)_{\mathscr{E}_p}$. The objects of this category $\mathcal{A}$ are the elementary abelian subgroups of G and the morphisms are the group homomorphisms $A \to B$ of the form $c_g : a \mapsto gag^{-1}$ for some $g \in G$.

To relate these two notions we construct a functor $J : \mathscr{O}(G)_{\mathscr{E}_p} \to \mathcal{A}$ sending G/A to A. Given a G-map $f : G/A_1 \to G/A_2$ satisfying $f(A_1) = gA_2$, we set $J(f) = c_{g^{-1}}$. Since the subgroups involved are abelian, this functor is well-defined.

Now J is a cofinal functor and hence the induced map $\lim_{\mathcal{A}^{\mathrm{op}}} F \to \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{E}_p}} J^*F$ is an isomorphism for every contravariant functor F indexed on $\mathcal{A}$. Now for a G-space X, the functor on $\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{E}_p}$ sending G/A to $H^*(EG \times_A X; \mathbb{F}_p)$ extends over J, so Quillen's limit is isomorphic to the one considered here.

However J is not *homotopy cofinal*; the higher limit terms are generally quite different. For example, in the case $G = Q_8$ just considered we have

$$\lim_{\mathcal{A}^{\mathrm{op}}}{}^{*} H^*(BA; \mathbb{F}_2) \cong \lim_{\mathcal{A}^{\mathrm{op}}}{}^{0} H^*(BA; \mathbb{F}_2) \cong H^*(BZ(Q_8); \mathbb{F}_2) \cong \mathbb{F}_2[e_3].$$

Since the higher limit functors vanish we see that the homotopy limit spectral sequence using Quillen's indexing category *will not* converge to $H^*(BQ_8; \mathbb{F}_2)$.

Example 5.22. We will now calculate the $\mathscr{F}$-exponent of Q_8-equivariant $\underline{H\mathbb{F}_2}$ for a slightly larger family than $\mathscr{E}_{(2)}$. Let f be one of the nontrivial classes in $H^1(Q_8; \mathbb{F}_2)$ and let σ be the pullback of the sign representation along f, so $\chi(\sigma) = f$. Now $f^3 = 0$ by the calculation above so $\underline{H\mathbb{F}_2}$ is a retract of $\underline{H\mathbb{F}_2} \wedge S(3\sigma)_+$. If we set $\mathscr{F}$ to be the family of subgroups contained in the kernel of f, then we see that the $\mathscr{F}$-exponent of $\underline{H\mathbb{F}_2}$ for $G = Q_8$ is at most 3. Moreover, $\exp_{\mathscr{F}}(\underline{H\mathbb{F}_2}) \geq 3$ because $f^2 \neq 0$ (cf. Remark 3.24), so we have in fact equality.

We now prove the integral version of the above result. We will frequently use the following.

Lemma 5.23. Fix a finite group G. For a spectrum E, we let $\mathscr{F}_E$ be the derived defect base of $\underline{E} \in \mathrm{Sp}_G$. If R is a ring spectrum, then $\mathscr{F}_R = \bigcup_{p||G|} \mathscr{F}_{R_p} = \bigcup_{p||G|} \mathscr{F}_{R_{(p)}}$ where R_p (resp. $R_{(p)}$) denotes the p-completion (resp. p-localization) of R.

Proof. We give the argument for the completions; the argument for the localizations is similar. Since $\underline{R_p}$ is an algebra over $\underline{R}$, we have $\mathscr{F}_R \supset \bigcup_p \mathscr{F}_{R_p}$. To obtain the opposite inclusion, we use the arithmetic square

$$\begin{array}{ccc} R & \longrightarrow & \prod_{p||G|} R_p \\ \downarrow & & \downarrow \\ R[1/|G|] & \longrightarrow & \left(\prod_{p||G|} R_p\right)[1/|G|]. \end{array} \tag{5.24}$$

This induces a pullback square upon taking Borel-equivariant theories. The Borel-equivariant forms of $R[1/|G|]$ and $\left(\prod_{p||G|} R_p\right)[1/|G|]$ have trivial derived defect base since they are $|G|^{-1}$-local (Theorem 4.25). As a result, we obtain $\mathscr{F}_R \subset \mathscr{F}_{\prod_{p||G|} R_p} = \bigcup_{p||G|} \mathscr{F}_{R_p}$ as desired. □

Proposition 5.25 (cf. [Car00])**.** The derived defect base of $\underline{H\mathbb{Z}}$ is $\mathscr{E}$.

We note that this result is essentially equivalent to [Car00, Thm. 2.1]. See also [Bal16, Thm. 4.3] for another equivalent statement stated in a language closer to ours.

Proof. We first prove that $\underline{H\mathbb{Z}}$ is $\mathscr{E}$-nilpotent. By Lemma 5.23, it suffices to show that $\underline{H\mathbb{Z}_p}$ is $\mathscr{E}_{(p)}$-nilpotent. Since $\underline{H\mathbb{Z}_p}$ is both an $\underline{MU}$-module and an $\underline{S_{(p)}}$-module, we already know that $\underline{H\mathbb{Z}_p}$ is $\mathscr{A}_{(p)}$-nilpotent by Theorems 5.14 and 4.25.

Moreover, $\underline{H\mathbb{Z}_p}$ has Thom isomorphisms for oriented representations. So by Proposition 5.4 it suffices to show that if A is an abelian p-group and $x \in H^*(BA; \mathbb{Z}_p)$ restricts to 0 on each elementary abelian subgroup, then x is nilpotent.

Suppose we have such an $x \in H^*(BA; \mathbb{Z}_p)$. Note that $|x|$ is necessarily greater than zero and, by Proposition 5.16, the mod-p reduction of x is nilpotent. In other words, a power of x is divisible by p. It follows that there exists $k \geq 1$ and $z \in H^*(BA;\mathbb{Z}_p)$ such that $x^k = |A|z$. Since $|A| \cdot H^*(BA;\mathbb{Z}_p) = 0$ for $* > 0$, we see that $x^k = 0$ as desired.

For the minimality claim, we note that since $H\mathbb{F}_p$ is an $H\mathbb{Z}$-module, the derived defect base of $\underline{H\mathbb{Z}}$ must contain $\mathscr{E}_{(p)}$ by Proposition 5.16. The claim now follows by varying p. □

5.5. **L_n-local Borel-equivariant theories.** Using Hopkins-Kuhn-Ravenel character theory, we now determine the minimal family for Borel-equivariant Morava E-theory and some related spectra.

Proposition 5.26 (cf. [GS99, HKR00])**.** Suppose that E is a complex-oriented homotopy commutative ring spectrum with associated formal group $\mathbb{G}$. Suppose further that:

- The coefficient ring π_*E is a complete local ring with maximal ideal m.
- The graded residue field π_*E/m has characteristic $p > 0$.
- The localization $\pi_*E[p^{-1}]$ is non-zero.
- The mod m reduction of $\mathbb{G}$ has height $n < \infty$.

Then the derived defect base of $\underline{E}$ is $\mathscr{A}^n_{(p)}$.

Proof. First we show that $\underline{E}$ is $\mathscr{A}^n_{(p)}$-nilpotent. Since E is complex oriented and p-local we already know $\underline{E}$ is $\mathscr{A} \cap \mathscr{A}\ell\ell_{(p)} = \mathscr{A}_{(p)}$-nilpotent. So by Proposition 5.4 it suffices to show that if A is an abelian p-group and $x \in E^*(BA)$ restricts to zero on $E^*(BA')$ for any $A' \leq A$ of rank $\leq n$, then x is nilpotent.

The results of [HKR00] show that, under the given hypotheses, there is a natural injection

$$E^*(BA) \hookrightarrow L(E^*) \otimes_{E^*} E^*(BA) \cong \hom_{\mathrm{Set}}(\mathrm{Gp}(\mathbb{Z}_p^n, A), L(E^*)) \tag{5.27}$$

of $E^*(BA)$ into a ring of generalized characters valued in some particular nontrivial $E^*[p^{-1}]$-algebra $L(E^*)$. By assumption, $x \in E^*(BA)$ is trivial on all of the subgroups of A which appear as images of some homomorphisms $\mathbb{Z}_p^n \to A$. It follows that x has trivial image in the character ring. Since $E^*(BA)$ injects into the character ring, x must be zero.

To see the minimality of this family, we will suppose that A is a product of n cyclic p-groups and find a non-nilpotent element $x \in E^*(BA)$ which restricts to zero on all proper subgroups As in [HKR00, Thm. C], there is an $\mathrm{Aut}(\mathbb{Z}_p^n)$-action on the right-hand-side of (5.27) such that $p^{-1}E^*(BA)$ is the $\mathrm{Aut}(\mathbb{Z}_p^n)$-invariants. Let $z \in \hom_{\mathrm{Set}}(\mathrm{Gp}(\mathbb{Z}_p^n, A), L(E^*))$ be the generalized character which sends each surjective homomorphism $\mathbb{Z}_p^n \to A$ to $1 \in L(E^*)$ and all other homomorphisms to zero. The element z is $\mathrm{Aut}(\mathbb{Z}_p^n)$-invariant and therefore belongs to $p^{-1}E^*(BA)$. Clearly z is idempotent and restricts to zero on all proper subgroups. Since the map in (5.27) is an isomorphism after inverting p, there is an $x \in E^*(BA)$ and a natural number k such that $p^k z = x$. By construction, x is non-nilpotent and restricts to zero on all proper subgroups.

□

The derived defect base shrinks if one quotients by an invariant ideal in $\pi_0 E$. For each positive integer n, let $\widehat{E}(n)$ denote the I_n-adically completed Johnson-Wilson theory. This is a complex oriented p-local cohomology theory whose coefficients $\pi_*\widehat{E}(n)$ are obtained by completing $\pi_*E(n) \cong \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$ at the ideal $I_n = (p, v_1 \ldots, v_{n-1})$ (here $v_0 = p$ conventionally).

Proposition 5.28. For $0 \leq k \leq n$, let $E = \widehat{E}(n)$ and $E/I_k = E/(p, v_1, \ldots, v_{k-1})$. The derived defect base of $\underline{E/I_k}$ is $\mathscr{A}^{n-k}_{(p)}$.

Proof. Using the v_n-periodicity of E it suffices to study the nilpotence of elements in degree 0. Let $\mathbb{G}$ be the formal group over $\pi_0(E/I_k)$ associated to the complex-oriented ring spectrum

E/I_k. Note that this is the reduction modulo I_k of the formal group associated to E. Let A be an abelian p-group and let $A^\vee$ denote the Pontryagin dual.

Recall that $\mathrm{Spec}\left((E/I_k)^0(BA)\right)$ is the formal scheme that classifies homomorphisms $A^\vee \to \mathbb{G}$. Since $E^0(BA)$ is a finite free module over $\pi_0 E$, we have

$$(E/I_k)^0(BA) \simeq \pi_0(E/I_k) \otimes_{\pi_0 E} E^0(BA).$$

By applying [GS99, Thm. 2.3] to E and then base-changing along $\pi_0 E \to \pi_0(E/I_k)$, one has a morphism of schemes

$$\coprod_{H \le A} \mathrm{Level}(H^\vee, \mathbb{G}) \to \mathrm{Spec}\left((E/I_k)^0(BA)\right),$$

which induces an isomorphism on underlying *reduced schemes.* Here $\mathrm{Level}(H^\vee, \mathbb{G})$ is the closed subscheme of $\mathrm{Spec}\left((E/I_k)^0(BH)\right)$ classifying *level structures* $H^\vee \to \mathbb{G}$ and the above map factors through the map induced by the restriction homomorphisms. Moreover, $\mathrm{Level}(H^\vee, \mathbb{G})$ is empty if and only if $\mathrm{rank}(H) > n - k$.

It follows now that the map of schemes

$$\coprod_{H < A} \mathrm{Spec}\left((E/I_k)^0(BH)\right) \to \mathrm{Spec}\left((E/I_k)^0(BA)\right) \tag{5.29}$$

is surjective on geometric points if and only if $\mathrm{rank}(A) > n - k$. If $\mathrm{rank}(A) > n - k$, it follows that any element in $(E/I_k)^0(BA)$ which restricts to zero on proper subgroups is nilpotent. This proves that the derived defect base of $\underline{E/I_k}$ is at most $\mathscr{A}_{(p)}^{n-k}$. Similarly, the analysis of (5.29) combined with Proposition 3.27 shows that the derived defect base can be no smaller. □

Example 5.30. We show explicitly that the derived defect base of $\underline{K(n)}$ is $\mathscr{T}$. Since $\mathscr{T}$ is the minimal family, we only need to show that these G-spectra are $\mathscr{T}$-nilpotent. Using Proposition 2.13, this can also be deduced from [GS96, Thm. 1.1] (i.e., the vanishing of Tate spectra).

Since $K(n)$ is complex orientable and p-local we already know that $\underline{K(n)}$ is $\mathscr{A} \cap \mathscr{A}\ell\ell_{(p)} = \mathscr{A}_{(p)}$-nilpotent. Now since $K(n)$ admits Thom isomorphisms for unitary representations, it suffices to show that if $A = C_{p^{i_1}} \times \cdots \times C_{p^{i_k}}$ is an arbitrary abelian p-group and $x \in K(n)^*(BA)$ restricts to zero on the trivial subgroup then x is nilpotent by Proposition 5.4. By the Künneth isomorphism for Morava K-theory and the well-known calculations of the complex-oriented cohomology of cyclic groups,

$$K(n)^*(BA) \cong K(n)^* \otimes \mathbb{F}_p[x_1, \ldots, x_k]/(x_1^{p^{i_1 n}}, \ldots, x_k^{p^{i_k n}})$$

and the kernel of the restriction map $\mathrm{Res}_e^A \colon K(n)^*(BA) \to K(n)^*(Be)$ is the ideal $(x_1, \ldots, x_k)$. This ideal is evidently nilpotent and hence so is x, proving the claim.

We can also obtain a variant for the telescopic replacement for $K(n)$.

Proposition 5.31 (Cf. [Kuh04])**.** Let X be a type n-finite complex and let $T(n)$ be its v_n-periodic localization. Then the derived defect base of $\underline{T(n)}$ is $\mathscr{T}$.

Proof. The spectrum $T(n)$ is a module over the v_n-periodic localization of the type n, A_∞-ring spectrum $\mathrm{End}(X)$. So it suffices to consider the case $T(n) := \mathrm{End}(X)[v_n^{-1}]$. Since this spectrum is obtained from an A_∞-ring by inverting a central element [Rav92, Lem. 6.1.2] it is A_∞ [Lur14, §7.2.4]. Now by Proposition 2.13 it suffices to show that the associated Tate object $\widetilde{E\mathscr{T}} \wedge \underline{T(n)}$ is contractible. This is [Kuh04, Cor. 1.6]. □

5.6. **Hybrids of L_n-local theories and $H\mathbb{Z}$-algebras.** We now include examples of Borel-equivariant theories where there are two contributions to the derived defect base: an L_n-local piece and a $H\mathbb{Z}$-algebra piece.

Proposition 5.32. The derived defect base of $\underline{BP\langle n\rangle}$ is $\mathscr{E}_{(p)} \cup \mathscr{A}^n_{(p)}$.

Proof. Since both $H\mathbb{F}_p$ and the completed Johnson-Wilson theory $\widehat{E}(n)$ are $BP\langle n\rangle$-modules, the minimality claim will follow from the minimality results for these module spectra proven in Propositions 5.16 and 5.26.

To show that $\underline{BP\langle n\rangle}$ is $\mathscr{E}_{(p)} \cup \mathscr{A}^n_{(p)}$-nilpotent, we argue by induction on n. The base case $n = 0$ follows from Proposition 5.25. So suppose $n > 0$. Since $\underline{BP\langle n\rangle}$ has Thom isomorphisms for unitary representations, by Proposition 5.4 it suffices to show that if $x \in BP\langle n\rangle^*(BG)$ restricts to zero in $BP\langle n\rangle^*(BA)$ for each $A \in \mathscr{E}_{(p)} \cup \mathscr{A}^n_{(p)}$ then x is nilpotent.

First observe that x maps to a nilpotent class in $(L_n BP\langle n\rangle)^*(BG)$ by Proposition 5.36 below. So by raising x to a power, we may assume that x is already zero in $(L_n BP\langle n\rangle)^*(BG)$. Moreover, by the inductive assumption, and raising x to a sufficiently high power, we may assume that x maps to zero under r in the long exact sequence

$$\cdots \to (BP\langle n\rangle)^{*-|v_n|}(BG) \xrightarrow{v_n} (BP\langle n\rangle)^*(BG) \xrightarrow{r} (BP\langle n-1\rangle)^*(BG) \to \cdots .$$

This means that $x = v_n y$ for some $y \in (BP\langle n\rangle)^*(BG)$.

The L_n-localization map fits into the following fiber sequence of $BP\langle n\rangle$-modules

$$\Gamma_n BP\langle n\rangle \to BP\langle n\rangle \to L_n BP\langle n\rangle.$$

Mapping BG into this sequence we obtain another fiber sequence of $BP\langle n\rangle$-modules

$$F(BG_+, \Gamma_n BP\langle n\rangle) \to F(BG_+, BP\langle n\rangle) \to F(BG_+, L_n BP\langle n\rangle).$$

By the long exact sequence in homotopy, x lifts to $(\Gamma_n BP\langle n\rangle)^*(BG)$. Now by [GM95a, Thm. 2.3, §3, and Thm. 6.1] we see that $\Gamma_n BP\langle n\rangle$ and hence $F(BG_+, \Gamma_n BP\langle n\rangle)$ are bounded above $BP\langle n\rangle$-modules. It follows that there is a power of v_n such that

$$v_n^r x = 0 \in (\Gamma_n BP\langle n\rangle)^*(BG), \quad r \gg 0.$$

Examining the long exact sequence we see that $0 = v_n^r x \in BP\langle n\rangle^*(BG)$. Moreover, since $x = v_n y$,

$$x^{r+1} = (v_n y)^{r+1} = v_n^r x y^r = 0$$

as desired. □

The key properties of $BP\langle n\rangle$ that are used above are that it is a ring spectrum with the desired homotopy groups, that $BP\langle n\rangle \to L_n BP\langle n\rangle$ is an equivalence on connective covers, and that it admits the standard cofiber sequence relating $BP\langle n\rangle$ to $BP\langle n-1\rangle$. As such, the argument is quite robust. We give another example of this argument below.

Proposition 5.33. Let R be a connective E_∞-ring. Suppose that $\pi_*(R) \simeq \pi_0(R)[x_1, \ldots, x_n]$, where $x_i \in \pi_*(R)$ is in positive even degrees. Consider the finite localization R' of R away from $(x_1, \ldots, x_n)$ (cf. [GM95a]). For a spectrum X, let $\mathcal{F}_X$ denote the derived defect base of $\underline{X}$ with respect to a finite group G. Then we have

$$\mathcal{F}_R = \mathcal{F}_{R'} \cup \mathcal{F}_{H\pi_0 R}.$$

Proof. Since $R'$and $H\pi_0 R$ are R-algebras, the inclusion $\mathcal{F}_R \supset \mathcal{F}_{R'} \cup \mathcal{F}_{H\pi_0 R}$ is evident.

Let G be a finite group such that $\underline{R'}$, $\underline{H\pi_0 R} \in \mathrm{Sp}_G$ are nilpotent for the family of proper subgroups. It suffices to show that $\underline{R}$ is too. We will show that $\underline{R/(x_1, \ldots, x_k)}$ is nilpotent for the family of proper subgroups by descending induction on k. When $k = n$, this iterated cofiber is $H\pi_0 R$ and the induction hypothesis holds by assumption.

Suppose now that $\underline{R/(x_1,\dots,x_{k+1})}$ is nilpotent for the family of proper subgroups. We want to prove the analog with $k+1$ replaced by k. Note that each $R/(x_1,\dots,x_i)$ admits an A_∞-algebra structure in R-modules by [Ang08, Cor. 3.2]. Since $\pi_*(R)$ is concentrated in even degrees, R is complex orientable. It therefore suffices to show that if

$$u \in (R/(x_1,\dots,x_k))^*(BG)$$

restricts to zero on proper subgroups, it is nilpotent. The inductive hypothesis shows that a power u^k of u is a multiple of x_{k+1}, so it suffices to show that some power of u is annihilated by a power of x_{k+1}.

Let $\Gamma_n R$ denote the fiber of $R \to R'$, so that $\Gamma_n R$ has bounded-above homotopy groups via the spectral sequence of [GM95a, (3.2)]. We consider similarly the cofiber sequence

$$\Gamma_n R/(x_1,\dots,x_k) \to R/(x_1,\dots,x_k) \to R'/(x_1,\dots,x_k),$$

where $\Gamma_n R/(x_1,\dots,x_k)$ has bounded-above homotopy groups. Replacing u by a power of itself, we may assume that u maps to zero in $(R'/(x_1,\dots,x_k))^*(BG)$, so that it lifts to the module $(\Gamma_n R/(x_1,\dots,x_k))^*(BG)$. However, we see as before that every element of this (as a bounded above object) is annihilated by a power of x_{k+1}.

□

Corollary 5.34. The derived defect base of $\underline{ku}$ is $\mathscr{E} \cup \underline{\mathscr{C}}$.

Proof. By Lemma 5.23, it suffices to check the derived defect base of $\underline{ku_{(p)}}$ is $\mathscr{E}_{(p)} \cup \mathscr{C}_{(p)}$ for each prime p dividing the group order. Now since $ku_{(p)}$ splits as a wedge of suspensions of $BP\langle 1\rangle$ and $BP\langle 1\rangle$ is a $ku_{(p)}$-module, the derived defect base of $\underline{ku_{(p)}}$ is the derived defect base of $\underline{BP\langle 1\rangle}$. The claim now follows from Proposition 5.32.

□

Proposition 5.35. The derived defect base of $\underline{k(n)}$ is $\mathcal{E}_{(p)}$.

Proof. This is deduced similarly from the derived defect bases of $\underline{K(n)}$ and $\underline{H\mathbb{F}_p}$. We leave the details to the reader. □

5.7. **Thick subcategory arguments.** We will now show how to apply thick subcategory arguments, e.g. the thick subcategory theorem of Hopkins-Smith [HS98, Thm. 7], to extend the results of the previous section to non-orientable Borel-equivariant theories such as $\underline{tmf}$ and $\underline{L_n S^0}$.

Proposition 5.36. The derived defect base of $\underline{L_n S^0}$ is $\mathscr{A}^n_{(p)}$.

Proof. By the Hopkins-Ravenel smash product theorem, there exists $k \geq 0$ such that $L_n S^0$ is a retract of $\mathrm{Tot}_k E_n^{\wedge \bullet+1}$, the kth stage of the E_n-cobar construction [Rav92, §8]. Since $\underline{E_n}$ is $\mathscr{A}^n_{(p)}$-nilpotent (Proposition 5.26), for each positive integer k, the module spectrum $\underline{E_n^{\wedge k}}$ is $\mathscr{A}^n_{(p)}$-nilpotent. Taking finite homotopy limits, we see that the Borel-equivariant theories $\underline{\mathrm{Tot}_k E_n^{\wedge \bullet+1}}$ associated to the finite stages of the E_n-cobar construction are $\mathscr{A}^n_{(p)}$-nilpotent. Finally, since $\mathscr{A}^{n,\mathrm{Nil}}_{(p)}$ is closed under retracts, we see that $\underline{L_n S^0}$ is $\mathscr{A}^n_{(p)}$-nilpotent. Conversely, since $\underline{E_n}$ is an $\underline{L_n S^0}$-module, the minimality claim follows from Proposition 5.26. □

Lemma 5.37. Suppose that p is a prime and X is a p-local finite spectrum of type zero, i.e., $H_*(X;\mathbb{Q}) \neq 0$. Then $\underline{X \wedge M}$ is $\mathscr{F}$-nilpotent if and only if $\underline{M_{(p)}}$ is $\mathscr{F}$-nilpotent.

Proof. Note that the functor $X \mapsto \underline{X}$ preserves finite limits and colimits, so that $\underline{X \wedge M} \simeq i_* X \wedge \underline{M}$ for a finite spectrum X. It thus follows that the thick subcategories of Sp_G generated by $\underline{X \wedge M}$ and $\underline{M_{(p)}}$ are equal, so their derived defect bases are equal. □

Proposition 5.38. The derived defect base of $\underline{KO}$ is $\underline{\mathscr{C}}$, while the derived defect base of $\underline{ko}$ is $\underline{\mathscr{C}} \cup \mathscr{E}$.

Proof. Both of these statements are consequences of Wood's theorem [Mat, Thm. 3.2] which gives equivalences $C\eta \wedge KO \simeq KU$ and $C\eta \wedge ko \simeq ku$. Since $H_*(C\eta;\mathbb{Q}) \neq 0$ we can apply Lemma 5.37 and Proposition 5.6 and Corollary 5.34 to see that $\underline{KO}$ is $\underline{\mathscr{C}}$-nilpotent and $\underline{ko}$ is $\underline{\mathscr{C}} \cup \mathscr{E}$-nilpotent. The minimality of these families follows from the minimality results in Proposition 5.6 and Corollary 5.34 for their respective module spectra $\underline{KU}$ and $\underline{ku}$. □

Definition 5.39. Let $\mathcal{O}^{\mathrm{top}}$ be the Goerss-Hopkins-Miller sheaf of E_∞-ring spectra on $\overline{\mathcal{M}}_{ell}$, the compactified moduli stack of elliptic curves (cf. [Beh12]). Let $\mathcal{M}_{ell} \subset \overline{\mathcal{M}}_{ell}$ denote the locus parametrizing smooth elliptic curves.

- Let $TMF := \Gamma(\mathcal{M}_{ell}; \mathcal{O}^{\mathrm{top}})$ denote the derived global sections of $\mathcal{O}^{\mathrm{top}}$ over $\mathcal{M}_{ell}$.
- Let $Tmf := \Gamma(\overline{\mathcal{M}}_{ell}; \mathcal{O}^{\mathrm{top}})$ denote the derived global sections of $\mathcal{O}^{\mathrm{top}}$.
- Let tmf denote the connective cover of Tmf.

Note that by construction, we have a sequence

$$tmf \to Tmf \to TMF$$

of E_∞-ring maps, where the first map is the connective covering and the second map is induced by the restriction map on structure sheaves.

Proposition 5.40. The family $\underline{\mathscr{A}^2}$ is the derived defect base of both $\underline{Tmf}$ and $\underline{TMF}$.

Proof. To show that the derived defect bases of $\underline{Tmf}$ and $\underline{TMF}$ must contain $\underline{\mathscr{A}^2}$, it suffices to show this is true for the $\underline{Tmf}$-module $\underline{TMF}$. We will do this by constructing, for every prime p, a map $Tmf \to \widehat{E}$ of ring spectra such that the derived defect base of $\underline{\widehat{E}}$ is $\mathscr{A}^2_{(p)}$. By varying p we see that the derived defect base of $\underline{Tmf}$ must contain all of $\underline{\mathscr{A}^2}$.

Fix a supersingular elliptic curve C over $\overline{\mathbb{F}}_p$.[7] It determines a geometric point $x\colon \mathrm{Spec}(\overline{\mathbb{F}}_p) \to \mathcal{M}_{ell}$. Choose an affine étale neighborhood $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ of x. Note that R is finitely generated over $\mathbb{Z}$ (hence noetherian) and torsion-free. Let E denote the localization of $\mathcal{O}^{\mathrm{top}}(R)$ at the prime ideal corresponding to the point x. There is a canonical map of E_∞-rings $TMF \to E$. Furthermore, E is even periodic with trivial π_1, hence complex orientable, $\pi_0 E$ is a local ring, and the corresponding formal group $\mathbb{G}$ on $\pi_0 E \simeq \mathcal{O}_{\mathcal{M}_{ell},x}$ is the base-change of the formal group of the elliptic curve. In particular, the reduction of $\mathbb{G}$ modulo the maximal ideal of $\pi_0 E$ is the formal group of C, hence of height 2. It now follows from Proposition 5.26 applied to the completion [Lur11, §4] $\widehat{E}$ of E at the maximal ideal that the derived defect base of $\underline{\widehat{E}}$ is $\mathscr{A}^2_{(p)}$, as desired.

We will now prove that $\underline{Tmf}$ is $\underline{\mathscr{A}^2}$-nilpotent; the claim for $\underline{TMF}$ will then follow by the module structure. By Lemma 5.23, it suffices to show $\underline{Tmf_{(p)}}$ is $\mathscr{A}^2_{(p)}$-nilpotent for each prime p (dividing $|G|$). Since $Tmf_{(p)}$ is L_2-local [Beh12], and hence an L_2S^0-module, the result now follows from Proposition 5.36. □

Proposition 5.41. The derived defect base of $\underline{tmf}$ is $\underline{\mathscr{A}^2} \cup \mathscr{E}$.

Proof. For the minimality claim, we note that $H\mathbb{Z}$ is a tmf-module and hence the derived defect base of $\underline{tmf}$ must contain $\mathscr{E}$ by Proposition 5.25. Since Tmf is also a tmf-module, the derived defect base of $\underline{tmf}$ must also contain $\underline{\mathscr{A}^2}$ by Proposition 5.40.

To prove that $\underline{tmf}$ is $\underline{\mathscr{A}^2} \cup \mathscr{E}$-nilpotent, we will use Lemma 5.23 and check this locally at every prime.

[7]The existence of such a curve for every p is classical, it follows from the Eichler-Deuring mass formula [Sil86, Exer. V.5.9].

(1) At the prime 2 we recall that there is a finite 2-local spectrum $DA(1)$ of type zero such that $DA(1) \wedge tmf_{(2)} \simeq BP\langle 2\rangle$ [Mat, Thm. 5.8]. It now follows from Proposition 5.32 and Lemma 5.37 that $\underline{tmf_{(2)}}$ is $\mathscr{E}_{(2)} \cup \mathscr{A}^2_{(2)}$-nilpotent.

(2) A similar argument at the prime 3 applies using a finite 3-local complex F of type zero such that $F \wedge tmf_{(3)} \simeq tmf_1(2)_{(3)}$ (cf. [Mat, Thm. 4.13]). One now applies Proposition 5.33 to determine the derived defect base of $\underline{tmf_1(2)_{(3)}}$ (whose homotopy groups are polynomial on classes a_2, a_4 in π_4, π_8) and hence that of $\underline{tmf_{(3)}}$. Note that the finite localization of $tmf_1(2)_{(3)}$ away from the ideal (a_2, a_4) is $Tmf_1(2)_{(3)}$ because the compactified moduli stack $(M_{\overline{ell},1}(2))_{(3)}$ is $(\mathrm{Spec}\mathbb{Z}_{(3)}[a_2,a_4] \setminus V(a_2,a_4))/\mathbb{G}_m$. This is in particular L_2-local by construction, so that the Borel-equivariant theory $\underline{Tmf_1(2)_{(3)}}$ is $\mathscr{A}^2$-nilpotent as before.

(3) At $p \geq 5$, one applies Proposition 5.33 directly to $tmf_{(p)}$, which is now complex orientable and whose homotopy groups are $\mathbb{Z}_{(p)}[c_4, c_6]$. Similarly, the finite localization away from the ideal generated by (c_4, c_6) is $Tmf_{(p)}$ and is therefore L_2-local by construction. Therefore, one can conclude as before.

□

5.8. **Additional bordism theories.** Finally, we determine the derived defect bases for the Borel-equivariant forms of a few additional bordism theories.

Proposition 5.42. The derived defect base of $\underline{MSO}$ is $\mathscr{E}_{(2)} \cup \mathscr{A}[1/2]$.

Proof. By Lemma 5.23, it suffices to show that the derived defect bases of $\underline{MSO_{(2)}}$ and $\underline{MSO[1/2]}$ are $\mathscr{E}_{(2)}$ and $\mathscr{A}[1/2]$, respectively.

Using a result of Wall [Wal60, Thm. 5], $MSO_{(2)}$ admits an $H\mathbb{Z}_{(2)}$-module structure [Sto68, p. 209] and hence $\underline{MSO_{(2)}}$ is $\mathscr{E}_{(2)}$-nilpotent by Proposition 5.25. This family is minimal since $H\mathbb{Z}_{(2)}$ is an $MSO_{(2)}$-algebra via the zeroth Postnikov section.

It is well known that the evident composite

$$MSp \to MU \to MSO \tag{5.43}$$

of ring maps induces an isomorphism in $\mathbb{Z}[1/2]$-homology. Since these spectra are connective the composite in (5.43) is a homotopy equivalence after inverting 2. It follows that the derived defect base of $\underline{MU[1/2]}$ is bounded above by the derived defect base for $\underline{MSp[1/2]}$ and bounded below by the derived defect base for $\underline{MSO[1/2]}$ and that all of these derived defect bases are equal. Now, by Theorems 4.25 and 5.14, each of these derived defect bases is $\mathscr{A}\ell\ell[1/2] \cap \mathscr{A} = \mathscr{A}[1/2]$. □

As shown in the course of the previous proof we have:

Corollary 5.44. The derived defect base of $\underline{MSp[1/2]}$ is $\mathscr{A}[1/2]$.

In general, the map of ring spectra $MSp \to MU$ induced by forgetting structure and Theorem 5.14 show that any defect base of $\underline{MSp}$ must contain $\mathscr{A}$.

Proposition 5.45. The derived defect base of $\underline{MSpin}$ is $\mathscr{C}_{(2)} \cup \mathscr{E}_{(2)} \cup \mathscr{A}[1/2]$.

Proof. By Lemma 5.23, it suffices to show that the derived defect bases of $\underline{MSpin[1/2]}$ and $\underline{MSpin_{(2)}}$ are $\mathscr{A}[1/2]$ and $\mathscr{C}_{(2)} \cup \mathscr{E}_{(2)}$ respectively.

For the first part, we will show that the forgetful map

$$f \colon MSpin \to MSO$$

is an equivalence after inverting 2, and hence the derived defect base of $\underline{MSpin[1/2]}$ is $\mathscr{A}[1/2]$, the derived defect base of $\underline{MSO[1/2]}$ (Proposition 5.42). Since these spectra are connective it suffices to show that f induces an isomorphism in $\mathbb{Z}[1/2]$-homology. By applying the Thom isomorphism, it then suffices to show that the forgetful map

$$BSpin \to BSO$$

induces an isomorphism in $\mathbb{Z}[1/2]$-homology. This last claim follows immediately from the collapse of the Serre spectral sequence:

$$H_*(BSO; H_*(B\mathbb{Z}/2; \mathbb{Z}[1/2])) \cong H_*(BSO; \mathbb{Z}[1/2]) \Longrightarrow H_*(BSpin; \mathbb{Z}[1/2]).$$

Now 2-locally $MSpin$ additively splits as a wedge of $ko_{(2)}$-modules [ABP67] and $ko_{(2)}$ is an $MSpin_{(2)}$-module via the Atiyah-Bott-Shapiro orientation. It follows that the derived defect base of $\underline{MSpin_{(2)}}$ is equal to the derived defect base of $\underline{ko_{(2)}}$ which is $\mathscr{C}_{(2)} \cup \mathscr{E}_{(2)}$ by Proposition 5.38 and Theorem 4.25. □

Proposition 5.46. The derived defect base of $\underline{MString[1/2]}$ is $\underline{\mathscr{A}}[1/2]$.

Proof. Since $MSpin$ is an $MString$-module via the forgetful map, any derived defect base of $\underline{MString[1/2]}$ must contain $\underline{\mathscr{A}}[1/2]$ by Proposition 5.45.

If p is a prime greater then 3, then $MString_{(p)}$ splits into a wedge of suspensions of BP [HR95, Cor. 2.2.3]. Hence $\underline{MString_{(p)}}$ is $\mathscr{A}_{(p)}$-nilpotent by Theorems 4.25 and 5.14. It now follows from Lemma 5.23 that $\underline{MString[1/6]}$ is $\underline{\mathscr{A}}[1/6]$-nilpotent. At the prime 3 there is a finite 3-local cell complex Y of type zero such that $Y \wedge MString$ splits into a wedge of BP's [HR95, Cor. 2.2.1]. So $\underline{MString_{(3)}}$ is $\mathscr{A}_{(3)}$-nilpotent by Lemma 5.37 and Theorem 5.14. We can now assemble these results with Lemma 5.23 to conclude that $\underline{MString[1/2]}$ is $\underline{\mathscr{A}}[1/2]$-nilpotent. □

It is an open problem to determine if there is an analogue of the Anderson-Brown-Peterson splitting of $MSpin_{(2)}$ for $MString_{(2)}$. If $MString_{(2)}$ split additively into a wedge of $tmf_{(2)}$-modules then our methods would show that the derived defect base of $MString$ is $\underline{\mathscr{A}}[1/2] \cup \mathscr{E}_{(2)} \cup \mathscr{A}^2_{(2)}$.

Proposition 5.47. The derived defect base of $\underline{MU\langle 6\rangle}$ is $\underline{\mathscr{A}}$. Similarly for $\underline{MSU}$.

Proof. It suffices to treat $MU\langle 6\rangle$, since MSU is between $MU\langle 6\rangle$ and MU. Since MU is an $MU\langle 6\rangle$-module via the forgetful map any derived defect base of $\underline{MU\langle 6\rangle}$ must contain $\underline{\mathscr{A}}$ by Theorem 5.14.

By [HR95, Cor. 2.2.3] $MU\langle 6\rangle_{(p)}$ splits into a wedge of BP's after localizing at any prime p greater than 3 and hence $\underline{MU\langle 6\rangle[1/6]}$ is $\underline{\mathscr{A}}[1/6]$-nilpotent. At the primes 2 and 3, there are finite p-local cell complexes of type zero such that smashing $MU\langle 6\rangle$ with either complex yields a wedge of BP's [HR95, Cor. 2.2.1 and Cor. 2.2.3]. Applying the thick subcategory theorem, Lemma 5.37, and Lemma 5.23 we now conclude that $\underline{MU\langle 6\rangle}$ is $\underline{\mathscr{A}}$-nilpotent. □

We close with a couple of open problems.

Problem 5.48. For a prime p, find a p-local ring spectrum R such that the derived defect base $\mathscr{F}$ of $\underline{R}$ satisfies

$$\mathscr{A}_{(p)} \subsetneq \mathscr{F} \subsetneq \mathscr{A}\ell\ell_{(p)}$$

where all of the inclusions are proper.

A potential approach to the previous problem is to solve the following:

Problem 5.49. Find the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that the derived defect base of $\underline{MO\langle n\rangle}$ or $\underline{MU\langle n\rangle}$ properly contains $\underline{\mathscr{A}}$. Here we set

$$\underline{MO\langle \infty\rangle} := \operatorname{holim}_n \underline{MO\langle n\rangle} \simeq \underline{S}$$

and similarly for $\underline{MU\langle \infty\rangle}$.

The results above show that $n \geq 8$.

Complex orientability and thick subcategory arguments played a key role in our arguments. An answer to the next problem, which surely has been posed before, would provide a lower bound on the n in Problem 5.49.

Problem 5.50. Find the smallest $n \in \mathbb{N} \cup \{\infty\}$ such that there is a prime p where either $MO\langle n\rangle_{(p)}$ or $MU\langle n\rangle_{(p)}$ is not in the smallest thick subcategory containing $MU_{(p)}$ and its module spectra.

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Appendix A. A toolbox for calculations

Below we provide a few technical results for working with $\mathscr{F}$-homotopy (co)limit spectral sequences.

A.1. **The classifying space $E\mathscr{F}$.** We will now verify some claims about the classifying space $E\mathscr{F}$ which were used in the body of the paper.

Let $i\colon \mathscr{O}(G)_{\mathscr{F}} \to \mathcal{S}_G$ be the inclusion of the full subcategory spanned by the transitive G-sets with isotropy in $\mathscr{F}$. We have defined $E\mathscr{F}$ to be $\operatorname{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} i$. We can model this G-space by the standard two-sided bar construction [BK72, §XII.5][May96, §V.2]:

$$E\mathscr{F} := \operatorname*{hocolim}_{\mathscr{O}(G)_{\mathscr{F}}} i \simeq |B_\bullet(*, \mathscr{O}(G)_{\mathscr{F}}, i)|, \tag{A.1}$$

where $B_\bullet(*, \mathscr{O}(G)_{\mathscr{F}}, i)$ is the simplicial G-space which in degree n is

$$\coprod_{(G/H_0,\ldots,G/H_n)\in \mathscr{O}(G)_{\mathscr{F}}^{\times n+1}} * \times \mathscr{O}(G)_{\mathscr{F}}(G/H_n, G/H_{n-1}) \times \cdots \times \mathscr{O}(G)_{\mathscr{F}}(G/H_1, G/H_0) \times i(G/H_0).$$

The zeroth face map is the projection which sends $\mathscr{O}(G)_{\mathscr{F}}(G/H_n, G/H_{n-1})$ to a point and is the identity on the other factors. Using the functoriality of i we obtain a map

$$\mathscr{O}(G)_{\mathscr{F}}(G/H_1, G/H_0) \times i(G/H_0) \to i(G/H_1).$$

The last face map is the product of this with map with the identity on the remaining factors. The remaining face maps come from the composition in $\mathscr{O}(G)_{\mathscr{F}}$ and the degeneracies come from including identities into the hom-factors.

Proposition A.2. The G-space $E\mathscr{F}$ has the following properties:

(1) The fixed points of $E\mathscr{F}$ have the following homotopy types:

$$E\mathscr{F}^K \simeq \begin{cases} * & \text{if } K \in \mathscr{F} \\ \emptyset & \text{otherwise.} \end{cases}$$

(2) Let $\mathcal{S}_{\mathscr{F}} \subseteq \mathcal{S}_G$ denote the full subcategory spanned by those G-spaces which admit a G-CW structure with cells having isotropy only in $\mathscr{F}$. Then $E\mathscr{F}$ is a homotopically terminal object in $\mathcal{S}_{\mathscr{F}}$.

(3) The G-space $E\mathscr{F}$ is determined up to equivalence by Condition (1).

Proof. We only give the proof of the first assertion. Let $K \leq G$ be such that $K \notin \mathscr{F}$. Since K-fixed points commute with homotopy colimits, it follows easily that $(E\mathscr{F})^K = \emptyset$. Suppose now $K \in \mathscr{F}$; then we have

$$\underset{G/H \in \mathcal{O}_{\mathscr{F}}(G)}{\operatorname{hocolim}} (G/H)^K = \underset{G/H \in \mathcal{O}_{\mathscr{F}}(G)}{\operatorname{hocolim}} \hom_{\mathcal{S}_G}(G/K, G/H) \simeq *$$

because the homotopy colimit of a corepresentable functor is contractible. □

A.2. **Cofinality results.** The following cofinality results aid in the calculation of $\mathscr{F}$-homotopy (co)limit spectral sequences.

Lemma A.3. Let N be a normal subgroup of G. If $\mathscr{F}$ is the family of all subgroups of N then the inclusion $i\colon BG/N^{\mathrm{op}} \to \mathscr{O}(G)_{\mathscr{F}}$ is homotopy cofinal. In particular, the derived functors of colimits and limits over $\mathscr{O}(G)_{\mathscr{F}}$ for $\mathcal{F}$ the family of subgroups contained in N are identified with group (co)homology for G/N.

Proof. This is a special case of [MNN, Prop. 6.31]. □

Proposition A.4. Let p and q be two distinct primes and $\mathscr{F}_1$ (resp. $\mathscr{F}_2$) the family of p-subgroups (resp. q-subgroups) of the finite group G. Then the following commutative square of categories

$$\begin{array}{ccc} BG & \longrightarrow & \mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}_1} \\ \downarrow & & \downarrow \\ \mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}_2} & \longrightarrow & \mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}_1 \cup \mathscr{F}_2} \end{array} \tag{A.5}$$

induces a pushout of simplicial sets upon applying the nerve functor.

Proof. It suffices to prove the statement above for the opposite categories. That is, we show

$$N\mathscr{O}(G)_{\mathscr{F}_1 \cup \mathscr{F}_2} \cong N\mathscr{O}(G)_{\mathscr{F}_1} \cup_{NBG^{\mathrm{op}}} N\mathscr{O}(G)_{\mathscr{F}_2}$$

is the pushout of the nerves. Note that the pushout simplicial set $N\mathscr{O}(G)_{\mathscr{F}_1} \cup_{NBG^{\mathrm{op}}} N\mathscr{O}(G)_{\mathscr{F}_2}$ is just a set-theoretic union in each degree.

So we need to show that any n-simplex in the nerve of $\mathscr{O}(G)_{\mathscr{F}_1 \cup \mathscr{F}_2}$ lies entirely in the nerve of $\mathscr{O}(G)_{\mathscr{F}_1}$ or entirely in the nerve of $\mathscr{O}(G)_{\mathscr{F}_2}$; and if the n-simplex lies in both, then it must lie entirely in their intersection NBG^{op}. When n is 0, the n-simplices of a nerve correspond to the objects of the category and this claim is obvious. When n is positive the n-simplices correspond to chains of morphisms of length $(n-1)$.

Now suppose that H_p is a p-subgroup of G and H_q is a q-subgroup of G, then

$$\mathscr{O}(G)_{\mathscr{F}_1 \cup \mathscr{F}_2}(G/H_p, G/H_q) \neq \emptyset \iff H_p \text{ is subconjugate to } H_q \iff H_p = e.$$

This argument is symmetric in p and q, so any chain of morphisms in $\mathcal{O}(G)_{\mathcal{F}_1\cup\mathcal{F}_2}$ is either in the image of $\mathcal{O}(G)_{\mathcal{F}_1}$ or in the image of $\mathcal{O}(G)_{\mathcal{F}_2}$ under the embeddings in (A.5). If the chain of morphisms is in both categories, then it is a sequence of endomorphisms of G/e as desired. □

Proposition A.6. Let p and q be two distinct primes and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) the family of p-subgroups (resp. q-subgroups) of the finite group G. Let $\mathcal{C}$ be a complete ∞-category. Then for any functor

$$F\colon \mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_1\cup\mathcal{F}_2} \to \mathcal{C}$$

the decomposition in Proposition A.4 induces a homotopy pullback diagram in $\mathcal{C}$:

$$\begin{array}{ccc} \operatorname{holim}_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_1\cup\mathcal{F}_2}} F & \longrightarrow & \operatorname{holim}_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_1}} F|_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_1}} \\ \downarrow & & \downarrow \\ \operatorname{holim}_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_2}} F|_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{F}_2}} & \longrightarrow & \operatorname{holim}_{BG} F|_{BG} \end{array}$$

Proof. Applying the nerve functor to the pushout diagram from Proposition A.4 we obtain a pushout diagram of ∞-categories where, since the inclusions are fully faithful, each map is a monomorphism. The claim now follows from the dual of [Lur09, Prop. 4.4.2.2]. □

Appendix B. A sample calculation in equivariant K-theory

In this section we analyze the $\mathcal{C}$-homotopy limit spectral sequence converging to $\pi^G_* KU$ when $G = C_2 \times C_2$ is the Klein 4-group and $\mathcal{C} = \mathcal{P}$ is the family of all cyclic subgroups of G. Of course we know the target groups of this spectral sequence and we will use this knowledge below. Although this seems like a silly way to spend one's time, this calculation does illustrate some standard techniques for calculating derived functors and for evaluating differentials in these spectral sequences. Moreover, this determines the 'stable' portion of the $\mathcal{C}$-homotopy limit spectral sequence abutting to the homotopy groups of the Picard spectrum of the category of G-equivariant KU-modules (cf. [MS]). We hope to return to this topic later.

Even this most elementary case is still nontrivial. We will leave minor details to the reader.

We first fix some notation for the various subgroups and quotient groups:

$$\begin{aligned} H_1 &= C_2 \times e < G & F_1 &= G/H_1 \\ H_2 &= e \times C_2 < G & F_2 &= G/H_2 \\ H_3 &= \Delta(C_2) < G & F_3 &= G/H_3 \end{aligned}$$

The quotient maps induce ring homomorphisms $R(F_i) \to R(G)$ such that the induced map

$$R(F_1) \otimes R(F_2) \to R(G)$$

is an isomorphism. Let σ_i denote both the complex sign representation of $F_i \cong C_2$ and the representation of G obtained by pulling back along the quotient map.

The $\mathcal{C}$-homotopy limit spectral sequence for KU takes the following form:

$$\lim^s_{\mathcal{O}(G)^{\mathrm{op}}_{\mathcal{C}}} \pi^{(-)}_t KU \Longrightarrow \pi^G_{t-s} KU$$

The abutment is

$$R(G)[\beta^{\pm 1}] = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1\sigma_2\}[\beta^{\pm 1}]$$

where β is the Bott periodicity generator in degree 2 and $R(G)$ is the complex representation ring in degree 0. Since $\pi^{(-)}_* KU$ is 2-periodic with respect to this generator, the E_2-page is 2-periodic as well.

The map sending a virtual representation to its virtual dimension defines a map $R(-) \to \underline{\mathbb{Z}}$ of Green functors with kernel the augmentation ideal functor $I(-)$. Although this map does not split as Mackey functors, it does split as coefficient systems. From this splitting we obtain:

Proposition B.1. The E_2-term of the $\mathscr{C}$-homotopy limit spectral sequence has the following form:

$$\lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}} \pi_*^{(-)} KU \cong \lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}} (\underline{\mathbb{Z}})[\beta^{\pm}] \oplus \lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}} (I(-))[\beta^{\pm}]$$

To calculate these summands we will use the identification, for coefficient systems M,

$$\lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}} (M) \cong \mathrm{Ext}^*_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}(\underline{\mathbb{Z}}, M)$$

of Section 3.1. One could calculate this directly from the definition by taking a projective resolution of $\underline{\mathbb{Z}}$ in coefficient systems. We will instead use a less direct method that can be be applied to a wider class of problems.

We will perform the analogous calculation for various subfamilies $\mathscr{F} \subseteq \mathscr{C}$ of subgroups, starting with the trivial family and gradually working our way up. For such a family let $\mathbb{Z}[\mathscr{F}]$ be the coefficient system obtained by restricting $\underline{\mathbb{Z}}$ to $\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}$ and then left Kan extending to a functor on $\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}$. We then define the coefficient system $\mathbb{Z}[\tilde{\mathscr{F}}]$ by the following short exact sequence:

$$0 \to \mathbb{Z}[\mathscr{F}] \xrightarrow{i} \mathbb{Z}[\mathscr{C}] \xrightarrow{\pi} \mathbb{Z}[\tilde{\mathscr{F}}] \to 0$$

where i is the counit of the left Kan extension/restriction adjunction.

From this short exact sequence we obtain the following long exact sequence of Ext-groups:
(B.2)

$$\cdots \mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{C}}}(\mathbb{Z}[\mathscr{F}], M) \xleftarrow{i^*} \mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{C}}}(\mathbb{Z}[\mathscr{C}], M) \xleftarrow{\pi^*} \mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{C}}}(\mathbb{Z}[\tilde{\mathscr{F}}], M) \xleftarrow{\partial} \mathrm{Ext}^{s-1}_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{C}}}(\mathbb{Z}[\mathscr{F}], M) \cdots$$

Just as in the proof of Corollary 3.6 we have an adjunction isomorphism

$$\mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{C}}}(\mathbb{Z}[\mathscr{F}], M) \cong \mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}(\underline{\mathbb{Z}}, M).$$

We will use this isomorphism and the long exact sequence of (B.2) repeatedly to calculate the E_2-term in Proposition B.1 by gradually increasing the size of the family under consideration.

B.1. **The trivial family of subgroups.** We begin by considering the trivial family of subgroups. In this case $\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{F}}$ is the category with one object G/e and whose morphisms are the elements of G. The composition law is obtained from the group multiplication and a projective resolution of $\underline{\mathbb{Z}}$ in $\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}$ is just a projective resolution of the trivial module $\mathbb{Z}$ in $\mathbb{Z}[G]$-modules. Under this identification the free module $\mathbb{Z}[G]$ corresponds to the restriction of the projective functor $\mathbb{Z}\{\mathscr{O}(G)(-, G/e)\}$ to the trivial family. This leads easily to the following identification (when $\mathscr{F} = \mathscr{T} = \{e\}$)

$$\mathrm{Ext}^s_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}(\underline{\mathbb{Z}}, M) \cong H^s(G; M(G/e)).$$

In the case $M = I$, $I(G/e) = 0$ so these groups vanish. To simplify the notation we will write $I(H) := I(G/H)$ below. When $M = \underline{\mathbb{Z}}$ this is just the integral cohomology of G, which we will denote by $H^*(G; \mathbb{Z})$ throughout this section.

We will now recall the well-known calculation of $H^*(G; \mathbb{Z})$ in order to fix notation and to relate it to the cohomology of the subgroups H_i and the quotient groups F_i. We will use the Bockstein spectral sequence from the cohomology with $\mathbb{F}_2$-coefficients. Recall that $H^*(F_i; \mathbb{F}_2)$ is a polynomial algebra on a generator x_i in degree 1. This element supports a nontrivial Bockstein $\beta x_i = \mathrm{Sq}^1 x_i = x_i^2$. By the Künneth theorem the quotient maps induce an isomorphism

$$H^*(F_1; \mathbb{F}_2) \otimes H^*(F_2; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, x_2] \cong H^*(C_2 \times C_2; \mathbb{F}_2).$$

The Bockstein spectral sequence collapses at E_2. There is only simple 2-torsion and no exotic multiplicative extensions:

$$H^*(C_2 \times C_2; \mathbb{Z}) \cong \mathbb{Z}[y_1, y_2, z]/(2y_1, 2y_2, 2z, z^2 - y_1 y_2^2 - y_1^2 y_2).$$

Here $y_i = \beta x_i$ is in degree 2 and $z = \beta(x_1 x_2)$ is in degree 3.

B.2. **The nearly trivial family of subgroups.** We now consider the case $\mathscr{F} = \{e, H_i\}$. By the Yoneda lemma, we can identify a map of coefficient systems

$$f \colon \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, G/H)\} \to M$$

with an element $f \in M(G/H)$. In particular we obtain an augmentation map

$$\varepsilon \colon \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, G/H)\} \to \underline{\mathbb{Z}}$$

corresponding to the unit. Similarly, for every element in the Weyl group $g \in N_G H/H = \mathscr{O}(G)(G/H, G/H)$ we obtain a map

$$g \colon \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, G/H)\} \to \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, G/H)\}$$

Lemma B.3. Let $\underline{\mathbb{Z}}$ denote the constant $G = C_2 \times C_2$-Green functor at the integers restricted to the family $\mathscr{F} = \{e, H_i\}$. Let g be a generator of the quotient group $F_i = N_G H_i \cong C_2$. Then the following sequence of functors is exact:

$$0 \to \underline{\mathbb{Z}} \to \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, F_i)\} \xrightarrow{e+g} \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, F_i)\} \xrightarrow{e-g} \mathbb{Z}\{\mathscr{O}(G)_{\mathscr{F}}(-, F_i)\} \xrightarrow{\varepsilon} \underline{\mathbb{Z}} \to 0.$$

Before proceeding to the proof, we note that we can concatenate these exact sequences together to obtain a projective resolution of $\underline{\mathbb{Z}}$. This immediately yields:

Corollary B.4. For the family $\mathscr{F} = \{e, H_i\}$, we have the following identification:

$$\operatorname{Ext}^*_{\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}}(\underline{\mathbb{Z}}, M) \cong H^*(F_i; M(G/H_i)).$$

Proof of Lemma B.3. Although this is a special case of Lemma A.3, we include an alternative, and perhaps more explicit, argument.

Since kernels and cokernels in $\mathbb{Z}\mathscr{O}(G)_{\mathscr{F}}$ are calculated object-wise, the exactness of a sequence of natural transformations is equivalent to the exactness of the sequence of maps obtained by evaluating at $G/H_i = F_i$ and G/e. In both cases we obtain the beginning of the standard 2-periodic $\mathbb{Z}[F_i]$-resolution of the trivial module $\mathbb{Z}$. □

We will now calculate the terms in Corollary B.4 when M is either of the summands $\underline{\mathbb{Z}}$ or $I(-)$ of $R(-)$. From the discussion in Appendix B.1 we know that $H^*(F_i; \mathbb{Z}) \cong \mathbb{Z}[y_i]/(2y_i)$. The action of F_i on $I(H_i) = \mathbb{Z}\{1 - \overline{\sigma}_i\}$ is via the conjugation action on H_i which is trivial since G is abelian. Regarding $H^*(F_i; I(H_i))$ as a module over $H^*(F_i, \mathbb{Z})$, we obtain:

$$H^*(F_i; I(H_i)) \cong \mathbb{Z}[y_i]/(2y_i) \otimes (1 - \overline{\sigma}_i).$$

Here $\overline{\sigma}_i$ is the sign representation of H_i, not $F_i = G/H_i$, so σ_j restricts to $\overline{\sigma}_i$ if and only if j is *not* i.

Note that the relations $(1 - \overline{\sigma}_i)^2 = 2(1 - \overline{\sigma}_i)$ and $2y_i = 0$ force all products of positive degree elements in $H^*(F_i; I(H_i))$ to vanish.

We will need to understand the behavior of the restriction map induced by the natural transformation

$$i \colon \mathbb{Z}[\{e\}] \to \mathbb{Z}[\{e, H_i\}]$$

of coefficient systems. Topologically this corresponds to the nontrivial map $EG_+ \to E\{e, H_i\}_+ \simeq EF_{i+}$ of pointed G-spaces. Using the preferred models $EG = |G^{\bullet+1}|$ and $EF_i = |F_i^{\bullet+1}|$, we see that this map is induced by the quotient map $G \to F_i$. It follows that

$$i^* \colon H^*(F_i; \mathbb{Z}) \to H^*(G; \mathbb{Z})$$

is induced by the quotient $G \to F_i$. Of course

$$i^*\colon H^*(F_i; I(H_i)) \to H^*(G; I(e)) = 0$$

is the zero map.

B.3. **The family $\mathscr{C} = \mathscr{P}$.** We can now assemble the above results to calculate the E_2-term from Proposition B.1. The sum of the counit maps

$$\bigoplus_{i=1}^{3} \mathbb{Z}[\{e, H_i\}] \to \underline{\mathbb{Z}}$$

is evidently surjective and yields the following short exact sequence of coefficient systems:

$$0 \to \mathbb{Z}[\{e\}] \oplus \mathbb{Z}[\{e\}] \xrightarrow{j} \bigoplus_{i=1}^{3} \mathbb{Z}[\{e, H_i\}] \to \underline{\mathbb{Z}} = \mathbb{Z}[\{e, H_1, H_2, H_3\}] \to 0. \tag{B.5}$$

Here j is the inclusion of the kernel, which is adjoint to the linear map between trivial $\mathbb{Z}[G]$-modules given by the following matrix:

$$j = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The short exact sequence in (B.5) induces the following long exact sequence in Ext-groups:
(B.6)

$$\cdots \xleftarrow{\partial} \bigoplus_{i=1}^{2} H^*(C_2 \times C_2; \mathbb{Z}[\beta^{\pm}]) \xleftarrow{j^*} \bigoplus_{i=1}^{3} H^*(F_i; R(H_i)[\beta^{\pm}]) \leftarrow H^*_{C_2 \times C_2}(E\mathscr{C}; \pi_*^{(-)} KU) \xleftarrow{\partial} \cdots.$$

Remark B.7. We can find G-spectra whose integral Bredon homology realizes the short exact sequence of coefficient systems in (B.5). Moreover, we can also lift the maps of a coefficient systems to maps of G-spectra:

$$\Sigma^\infty_+ EG \vee \Sigma^\infty_+ EG \xrightarrow{j} \bigvee_{i=1}^{3} \Sigma^\infty_+ E\mathscr{C}(H_i) \to \Sigma^\infty_+ E\mathscr{C}. \tag{B.8}$$

Mapping this sequence into KU and taking the associated Atiyah-Hirzebruch spectral sequences one can see that the maps in (B.5) are morphisms between the E_2-terms of these spectral sequences.

Theorem B.9. The E_2-term from Proposition B.1 can be explicitly identified as follows:

$$\lim^*_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}} (\pi_*^{(-)} KU) \cong \mathbb{Z}[\beta^{\pm}] \otimes (A \oplus B)$$

where

$$A = \mathrm{Im}\partial = \bigoplus_{i=1}^{2} \tilde{H}^{*-1}(C_2 \times C_2; \mathbb{Z}) / \left(\mathbb{Z}/2\{(y_1^k, 0), (y_2^k, y_2^k), (0, y_3^k)\}_{k \geq 1}\right) \tag{B.10}$$

and

$$B = \ker j^* = \mathbb{Z} \oplus \left(\bigoplus_{i=1}^{3} H^*(F_i; I(H_i)) \right). \tag{B.11}$$

Proof. Plugging the calculations of the previous sections into the associated long exact sequence from (B.2) we obtain an exact sequence

$$0 \to A \to \lim^*_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{C}}} R(-) \to B \to 0.$$

In the zeroth cohomological degree all of the terms are in B, so we have to check this sequence splits in positive degrees. In positive degrees the splitting of the coefficient system $R(-) \cong \mathbb{Z} \oplus I(-)$ splits this sequence. □

Remark B.12. We will now perform some simple consistency checks. Note that all of the positive filtration terms in this spectral sequence are 2-torsion in accordance with Proposition 3.12. One can independently verify the correctness of the 0-line by analyzing the representation rings.

Finally we note that if we restrict to any proper subgroup of $C_2 \times C_2$, then for some i and all positive k, y_i^k is sent to zero as are all the terms divisible by the z classes. The terms $I(H_i)$ restrict to zero on all of the subgroups except H_i, in which case the higher cohomology groups map to zero. It follows that all of the positive degree terms restrict to zero on any proper subgroup as expected.

B.4. **Analysis of the $\mathscr{C}$-homotopy limit spectral sequence.** In this section we will complete this calculation and prove:

Theorem B.13. Let $G = C_2 \times C_2$. The $\mathscr{C}$-homotopy limit spectral sequence

$$E_2^{s,t} = \lim^s_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{C}}} (\pi_t^{(-)} KU) \Longrightarrow \pi^G_{t-s} KU \cong R(G)[\beta^{\pm}]$$

collapses at E_4 onto the zero line. Moreover the E_2-edge homomorphism

$$R(G) \to \lim_{\mathcal{O}(G)^{\mathrm{op}}_{\mathscr{C}}} R(C)$$

is injective with cokernel $\mathbb{Z}/2$. A generator of the cokernel supports a nontrivial d_3.

We will break up the analysis of this spectral sequence using the splitting in Theorem B.9. To analyze the A-summand in (B.10) we will first determine the behavior of the classical Atiyah-Hirzebruch spectral sequence

$$H^s(G; \mathbb{Z}[\beta^{\pm}]) \cong \mathbb{Z}[y_1, y_2, z]/(2y_1, 2y_2, 2z, z^2 - y_1^2 y_2 - y_1 y_2^2)[\beta^{\pm}] \tag{B.14}$$

$$\Longrightarrow KU^{s-t}(BG) \cong \pi^G_{t-s} F(EG_+; KU). \tag{B.15}$$

This spectral sequence arises from a multiplicative filtration on the ring spectrum $R := F(EG_+, KU)$, which is compatible with the similarly defined filtration on the free module

$$\Sigma^{-1} R \vee \Sigma^{-1} R \simeq \Sigma^{-1}(R \times R) \simeq \Sigma^{-1} F(EG_+ \vee EG_+, KU) \simeq F(\Sigma(EG_+ \vee EG_+), KU).$$

We can now identify the spectral sequence abutting to $\pi^G_{t-s} F(\Sigma(EG_+ \vee EG_+), KU)$ as two shifted copies of the spectral sequence in (B.15). As discussed in Remark B.7, the spectral sequence abutting to $\pi^G_{t-s} F(\Sigma(EG_+ \vee EG_+, KU)$ maps to the $\mathscr{C}$-homotopy limit spectral sequence and by Theorem B.9 the image of this morphism is the A-summand. Through this comparison we can determine the differentials emanating from the A-summand from the differentials in (B.15).

Now by [Ati61, Prop. 2.4], the first differential in (B.15) is a d_3 given by the operation $\mathrm{Sq}^3_{\mathbb{Z}}$. This operation is defined to be the following composition

$$\mathrm{Sq}^3_{\mathbb{Z}} \colon H\mathbb{Z} \xrightarrow{-\otimes \mathbb{Z}/2} H\mathbb{Z}/2 \xrightarrow{\mathrm{Sq}^2} \Sigma^2 H\mathbb{Z}/2 \xrightarrow{\beta_{\mathbb{Z}}} \Sigma^3 H\mathbb{Z}.$$

Here $\beta_{\mathbb{Z}}$ is the boundary map induced by the short exact sequence of coefficients:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

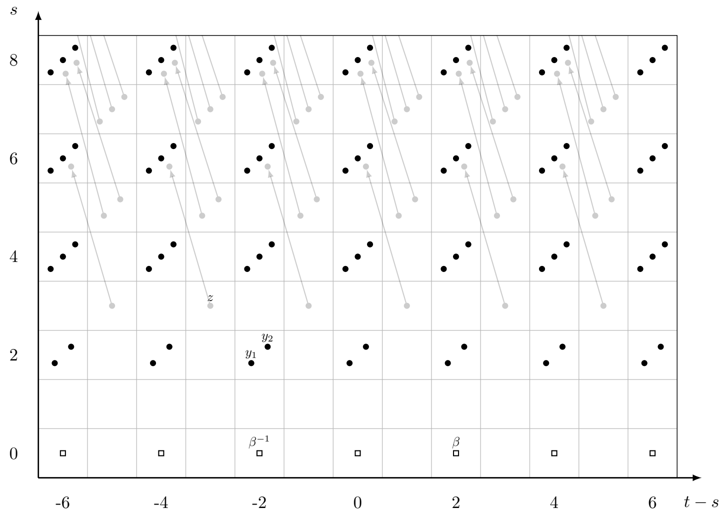

FIGURE B.16. The E_3-page of the Atiyah-Hirzebruch spectral sequence abutting to $\pi^G_{t-s}\underline{KU} \cong KU^{s-t}(BG)$, where $G = C_2 \times C_2$.

Now the mod-2 reduction of y_i^k is x_i^{2k} and an easy inductive argument shows that, for k odd, $\mathrm{Sq}^2 x_i^{2k} = x_i^{2k+2}$ and, for k even, $\mathrm{Sq}^2 x_i^{2k} = 0$. Now by our calculations from Appendix B.1, $\beta_{\mathbb{Z}}$ is zero on $x_i^{2\ell}$, for ℓ positive. It follows that $d_3(y_i^k) = \mathrm{Sq}^3_{\mathbb{Z}} y_i^k = 0$. Similarly

$$\mathrm{Sq}^3_{\mathbb{Z}} z = \beta_{\mathbb{Z}} \mathrm{Sq}^2(x_1^2 x_2 + x_1 x_2^2) = \beta_{\mathbb{Z}}(x_1^4 x_2 + x_1 x_2^4) = y_1^2 y_2 + y_1 y_2^2 = z^2,$$

so $d_3(z) = z^2$. Using the Leibniz rule, one generates all other differentials in this spectral sequence. The E_4-page is concentrated in even degrees, so the spectral sequence of Figure B.16 collapses at this stage.

Proof of Theorem B.13. Since the homotopy groups $\pi_*^{(-)} KU$ are concentrated in even degrees the first possible differential in the $\mathscr{C}$-homotopy limit spectral sequence is a d_3. We will first calculate this differential on the A-summand from (B.10):

$$\text{(B.17)} \qquad \partial\colon \mathbb{Z}[\beta^{\pm}] \otimes A = \bigoplus_{i=1}^{2} \tilde{H}^{*-1}(C_2 \times C_2; \mathbb{Z}[\beta^{\pm}]) \to H^*_{C_2\times C_2}(E\mathscr{C}; \pi_*^{(-)} KU).$$

Having determined the behavior of the spectral sequence in (B.15) we can now determine the differentials emanating from the A-summand. We see that $d_3(z,0) = (z^2, 0)$ and $d_3(0,z) = (0,z^2)$ and that these generate all d_3 differentials emanating from the A-term (see Figure B.18). Moreover all of the remaining classes from A are permanent cycles from E_4-onward, since they come from permanent cycles in the Atiyah-Hirzebruch spectral sequence abutting to $KU^*_G(\Sigma(EG_+ \vee EG_+))$.

Since the B-summand (B.11) is concentrated in even degrees, any possible d_3 emanating from it must land in the A-summand. Let us now calculate the differentials coming out of the zero line. An elementary analysis of the restriction maps $R(C_2 \times C_2) \to R(H_i)$ shows that the degree 0 part of the E_2-edge homomorphism

$$R(C_2 \times C_2) \to \lim_{\mathscr{O}(G)^{\mathrm{op}}_{\mathscr{C}}} R(-) \cong \mathbb{Z} \oplus \bigoplus_{i=1}^{3} \mathbb{Z}\{1 - \overline{\sigma}_i\}$$

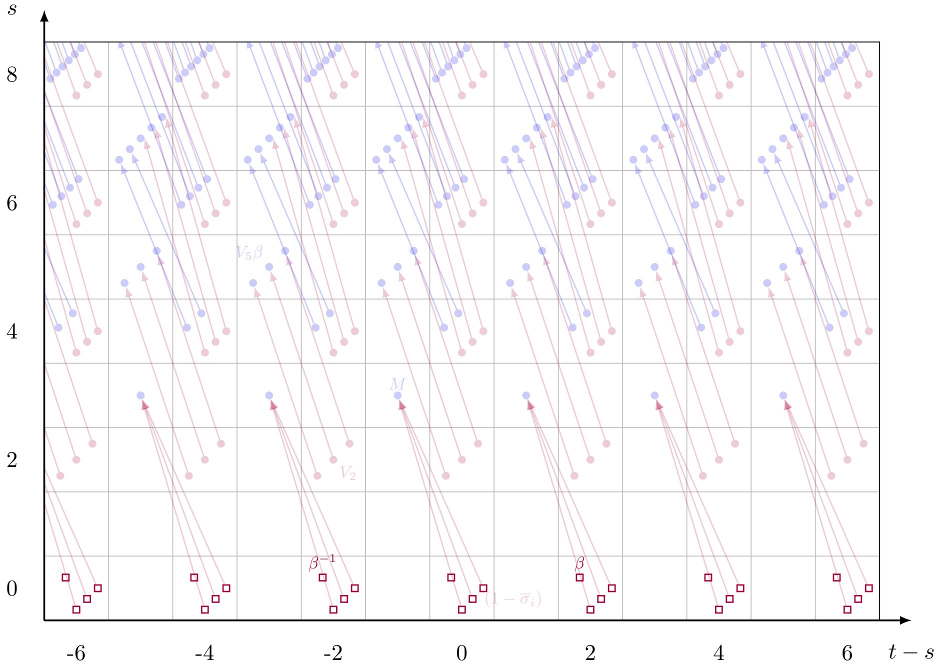

FIGURE B.18. The E_3-page of the $\mathscr{C}$-homotopy limit spectral sequence abutting to $\pi^G_{t-s}KU$, where $G = C_2 \times C_2$. The A-terms from (B.10) and the differentials emanating from them are tinted blue. The B-terms from (B.11) and the differentials emanating from them are tinted red.

sends the unit summand isomorphically to itself, while sending $(1-\sigma_i)$ to $\sum_{j\neq i}(1-\overline{\sigma}_j)$. It follows that the restriction map is injective with cokernel $\mathbb{Z}/2$ generated by Δ. We can choose $(1-\overline{\sigma}_i)$ as a generator of Δ for any i. Since the spectral sequence converges we know that that all terms in positive filtration must die and that Δ must support a differential, i.e., $d_i(\Delta) \neq 0$ for some $i \geq 3$.

We will now show that $d_3(\Delta) \neq 0$. Examining the E_2-term from Theorem B.9 we see that

$$d_3(\Delta) \in H^3_G(E\mathscr{C}; \pi_2^{(-)}KU) \cong \mathbb{Z}/2$$

which is generated by

$$M = (y_2, 0)\beta \equiv (y_3, y_3)\beta \equiv (0, y_1)\beta.$$

Now $M \in A$ is a permanent cycle. Since the positive filtration terms can not survive the spectral sequence, M must be hit by a differential emanating from the zero line. It follows that

$$d_3(\Delta) = d_3(1-\overline{\sigma}_i) = M$$

for each i.

The remaining terms in $E_4^{0,*}$ are the free abelian groups generated by

$$1, (1-\overline{\sigma}_1) + (1-\overline{\sigma}_3), (1-\overline{\sigma}_2) + (1-\overline{\sigma}_3), \text{ and } 2(1-\overline{\sigma}_3).$$

These are in the image of the restriction map and hence survive to the E_∞-page.

For the remaining terms we examine the Poincaré series for $H^*_G(E\mathscr{C}; R(-))$ (see Figures B.18 and B.19). We can argue inductively on the filtration degree to see that all of the A-terms which do not support a differential must be the target of a d_3 and that d_3 is injective on the positive

Term	Mod-$(2,\beta-1)$ Poincaré Series
A	$t\left(\frac{2(1+t^3)}{(1-t^2)^2}-\frac{3t^2}{1-t^2}-2\right)$
B	$1+\frac{3}{1-t^2}$
$d_3\Delta$	$1+t^3$
$d_3(z,0)y_1^*y_2^*$	$t\left(\frac{t^3(1+t^3)}{(1-t^2)^2}\right)$
$d_3(0,z)y_1^*y_2^*$	$t\left(\frac{t^3(1+t^3)}{(1-t^2)^2}\right)$
$d_3((1-\overline{\sigma}_i)y_i)y_i^*$	$\frac{t^2(1+t^3)}{1-t^2}$

FIGURE B.19. Poincaré series calculations. The Poincaré series of a differential is defined to be the series for the dimension of the image plus the dimension of the vector space mapping injectively to the image.

degree terms in B. For example, one can see that the 3-dimensional vector space $V_5\beta$ in filtration degree 5 must be in the image of a differential. Since the only possible differential out of the zero line is the d_3 we just calculated, we see that $V_5\beta$ must be the image of a d_3 coming from the 3-dimensional vector space

$$V_2 = \mathbb{F}_2\{(1-\overline{\sigma}_i)y_i\}_{1\leq i\leq 3}$$

in filtration degree 2. This pattern continues with d_3-differentials yielding isomorphisms between the remaining pairs of 3-dimensional vector spaces.

It follows that the $\mathscr{C}$-homotopy limit spectral sequence collapses at E_4 onto the zero line. □

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