Lifting homotopy T-algebra maps to strict maps

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Abstract

The settings for homotopical algebra—categories such as simplicial groups, simplicial rings, A_∞ spaces, E_∞ ring spectra, etc.—are often equivalent to categories of algebras over some monad or triple T. In such cases, T is acting on a nice simplicial model category in such a way that T descends to a monad on the homotopy category and defines a category of *homotopy* T-algebras. In this setting there is a forgetful functor from the homotopy category of T-algebras to the category of homotopy T-algebras.

Under suitable hypotheses we provide an obstruction theory, in the form of a Bousfield-Kan spectral sequence, for lifting a homotopy T-algebra map to a strict map of T-algebras. Once we have a map of T-algebras to serve as a basepoint, the spectral sequence computes the homotopy groups of the space of T-algebra maps and the edge homomorphism on π_0 is the aforementioned forgetful functor. We discuss a variety of settings in which the required hypotheses are satisfied, including monads arising from algebraic theories and operads. We also give sufficient conditions for the E_2-term to be calculable in terms of Quillen cohomology groups.

We provide worked examples in G-spaces, G-spectra, rational E_∞ algebras, and A_∞ algebras. Explicit calculations, connected to rational unstable homotopy theory, show that the forgetful functor from the homotopy category of E_∞ ring spectra to the category of H_∞ ring spectra is generally neither full nor faithful. We also apply a result of the second named author and Nick Kuhn to compute the homotopy type of the space $E_\infty(\Sigma^\infty_+ \operatorname{Coker} J, L_{K(2)}R)$.

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1. Introduction

In the work of Ando, Hopkins, Rezk, and Strickland on the Witten genus [6, 7, 5] the authors first construct a lift of the Witten genus to a multiplicative map of cohomology theories, then to an H_∞ map (i.e., a map preserving power operations), and finally to an E_∞ map

$$MString \to tmf.$$

In each of these steps they are asking that a map respects additional structure and it is natural to ask if there are general techniques for constructing such liftings.

Their construction of an H_∞ map makes use of ideas from Ando's thesis [3, 4], where he defines H_∞ maps from complex cobordism to Lubin-Tate spectra using a connection to isogenies of Lubin-Tate formal group laws. Their lift then arises from a computation: Since the H_∞ condition can be formulated in the stable homotopy category, a map is H_∞ if and only if an associated sequence of cohomological equations hold. The applicability of such techniques is one of the reasons that the category of H_∞ ring spectra is computationally more accessible. Although every E_∞ map forgets to an H_∞ map, constructing E_∞ maps is much more subtle and requires rather different techniques.

We construct an obstruction-theoretic spectral sequence to detect when an H_∞ map can be lifted to an E_∞ map and other problems of this type. As a consequence of our approach we can also see how much information is lost under the passage from E_∞ to H_∞ ring spectra. The first category can be described as the category of algebras over a monad/triple T in a category of spectra while the second is the category of such algebras in the homotopy category. Phrased in these terms, it is expected that a great deal is forgotten in the passage from E_∞ to H_∞ ring spectra. But to date, there have been no examples demonstrating this. Since our methods apply more generally to studying categories of algebras over a monad T (satisfying some hypotheses), we set up our machinery in the more abstract setting.

In Section 2 we provide a rapid review of the theory of monads and how they naturally encode algebraic structures. We emphasize the examples coming from algebraic theories and from operads since they make up the majority of our examples. In Section 3.1, we recall some conditions which guarantee the existence of a simplicial model structure on the category of algebras over a monad. These conditions are often satisfied and cover a broad range of standard examples. We include this standard material so the reader can easily apply it to the application of their choosing.

Our first main result is:

Theorem A. Let $\mathscr{C}$ be a simplicial model category and T a simplicial Quillen monad (Definition 3.1) acting on $\mathscr{C}$. Let X and Y be T-algebras. Suppose that

a. T commutes with geometric realization and
b. X is resolvable with bar cofibrant replacement $\widetilde{X} \to X$ (Definition 3.18).

Let $U\colon \mathscr{C}_T \to \mathscr{C}$ denote the forgetful functor from the category of T-algebras to $\mathscr{C}$. Then T induces a monad hT on $ho\mathscr{C}$ and there exists an obstruction-theoretic spectral sequence, called the T-algebra spectral sequence, such that:

1. Provided a T-algebra map $\varepsilon\colon X \to Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space

$$\pi^s \pi_t \mathscr{C}^d(T^\bullet U\widetilde{X}, UY) \Longrightarrow \pi_{t-s}\mathscr{C}^d_T(X,Y).$$

2. In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T-algebras.
3. The edge homomorphisms

$$\begin{aligned}\pi_0\mathscr{C}^d_T(X,Y) &\twoheadrightarrow E^{0,0}_\infty \\ &\hookrightarrow E^{0,0}_2 = (ho\mathscr{C})_{hT}(UX,UY) \\ &\hookrightarrow E^{0,0}_1 = ho\mathscr{C}(UX,UY)\end{aligned}$$

are the corresponding forgetful functors.

4. If $\mathscr{C}_T$ has functorial bar cofibrant and fibrant replacements, then the spectral sequence is contravariantly functorial in X and covariantly functorial in Y.
5. A map of simplicial monads $T_1 \to T_2$ satisfying the above hypotheses induces a contravariant map of spectral sequences provided that X has a bar cofibrant replacement $\widetilde{X} \to X$ in $\mathscr{C}_{T_2}$ such that $U_3\widetilde{X}$ has a bar cofibrant replacement in $\mathscr{C}_{T_1}$.

This result will be proven in Section 4.1. Note that since we avoid using E_2 model structures or Bousfield localizations, we do not require any properness assumptions on our model category. The assumption that our monad is simplicial Quillen is innocuous and satisfied in practice. The remaining two assumptions guarantee convergence to the desired target and allow us to identify the key terms in the spectral sequence.

Note that the convergence result in Theorem A is stronger than that of alternative approaches found in the literature, e.g., using a Reedy cofibrant replacement of the bar resolution or taking homotopy colimits instead of geometric realization. These approaches give spectral sequences which converge to mapping spaces from a T-cocompletion of the source, as in [22, 13]. We combine some standard results recalled in Section 3.2 with some crucial technical lemmas in Section 3.3 to prove convergence without a cocompletion under the assumptions of Theorem A.

In Section 4.4 we show that these assumptions hold in many general cases of interest such as nice categories of algebras over an operad, G-spaces and G-spectra (provided G is sufficiently nice), and many algebraic categories such as simplicial groups and rings. In each of these examples, the resolvability conditions hold for every object, so the spectral sequence can be applied to any pair of objects in the category.

This spectral sequence is a special case of the Bousfield-Kan spectral sequence. Bousfield has shown that this spectral sequence can be applied even without the existence of a base point—a useful generalization since a space of T-algebra maps may well be empty. In this case there is an obstruction theory (see Remark 4.4) for lifting a map in $\mathscr{C}$ to a map of T-algebras so that one can obtain a base point [16, § 5]. The farther one can lift this base point up the totalization tower, the greater the range in which one can define the spectral sequence and differentials.

As shown in Theorem 4.5, when the relevant mapping spaces in $\mathscr{C}$ have the homotopy type of H-spaces, e.g., if $\mathscr{C} = \mathcal{S}pectra$, then one can choose these obstructions to land in the E_2 page of the spectral sequence. Under favorable circumstances we can then apply our second main theorem, Theorem B of Section 4.3, to identify the E_2 term with Quillen cohomology groups.

We then demonstrate the wide applicability of this spectral sequence and its computability via Theorem B through a number of examples in Section 5. The reader interested in applications is encouraged to skip directly to this section where we compute the homotopy groups of particular:

- Spaces of equivariant maps in G-spaces and G-spectra (Section 5.1). This is a warm-up for the other examples. In two examples, we explicitly analyze the forgetful functor from the homotopy category of (strict) G-objects to (weak) G-objects in the homotopy category of spaces or spectra.
- Spaces of E_∞ maps between function spectra (Section 5.2). In two examples arising from unstable rational homotopy theory, we show that the forgetful functor from E_∞ to H_∞ ring spectra is generally neither full nor faithful. To the authors' knowledge, these are the first such examples.
- Spaces of A_∞ and E_∞ self-maps of Hk-algebras, for k a suitable field, whose homotopy rings are polynomial algebras.
- Spaces of E_∞ maps from $\Sigma^\infty_+ \operatorname{Coker} J$ to a $K(2)$-local E_∞ ring spectrum (Section 5.3). This is a result of Nick Kuhn and the second named author, and gives a nontrivial example of when the set of H_∞ maps coincides with the set of homotopy classes of E_∞ maps. As a consequence of the proof we obtain new information about $\operatorname{Coker} J$, including its $K(2)$-homology:

$$K(2)_* \operatorname{Coker} J \cong \bigoplus_{n\geq 0} K(2)_* B\Sigma_n.$$

Related work

The T-algebra spectral sequence arises by taking a functorial resolution of the source X. Namely we replace X by the two sided bar construction $B(F_T, T, UX)$ where U is the forgetful functor $\mathscr{C}_T \to \mathscr{C}$ and F_T is its left adjoint. For this approach, one wants general conditions under which the replacement is cofibrant, weakly equivalent to X, and equipped with a suitable filtration for obtaining a spectral sequence. A number of special cases of this theory are well known, and the arguments for spaces and spectra can be found in the literature. Although the two-sided bar construction has been a standard tool in homotopy theory for decades, we know of no reference in which its homotopical properties are developed with sufficient breadth for our purposes. In Section 3.3 we develop new tools for this purpose and apply them in Section 4.4 to demonstrate the applicability of the T-algebra spectral sequence.

There are a couple of alternative methods for constructing maps of structured ring spectra. This work can be considered an extension of the obstruction theory for maps of algebras in simplicial R-modules and A_∞ ring spectra that appears in Rezk's thesis [43] and his presentation of the Hopkins-Miller theorem [44] respectively. Indeed the latter work was a significant source of inspiration for this project. Angeltveit [8] has also constructed an obstruction theory, which appears to be part of a spectral sequence, for computing maps of A_∞ ring spectra.

The Goerss-Hopkins spectral sequence also computes the homotopy of the derived mapping space between two spectra which are algebras over a suitable operad, such as an E_∞ operad [19, 20]. This spectral sequence uses an E_2 model structure which guarantees an algebraic description of the E_2 term and is, in general, *not* the same as the T-algebra spectral sequence. In particular, their edge homomorphism is generally a Hurewicz homomorphism which usually is distinct from the forgetful functor above. In the sequel [39] however, the second named author shows that in special cases, such as the worked examples in Section 5.2, the spectral sequences do agree and computations can be done in either framework.

When Theorem B does not apply, it is generally quite difficult to determine the E_2 term of the T-algebra spectral sequence. Indeed, when T is the monad associated to the E_∞ operad, the main results of [7, 4, 27] could be expressed as partial computations of $d_1\colon E_1^{0,0} \to E_1^{1,0}$. The difficulties here are generic: there are very few examples where Theorem B does not apply yet one still has enough knowledge of the power operations to compute the E_2 term explicitly.

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Conventions / Terminology

We will make the convention that a simplicial category is a simplicially enriched category which is tensored and cotensored over simplicial sets. This convention is standard when discussing simplicial model categories, but unusual in enriched category theory.

2. Algebras over a monad

This section reviews monads and their categories of algebras, focusing on examples and conditions which ensure that limits and colimits in the categories of algebras exist. The existence of these constructions is not automatic, but will be essential for the material in Section 3. We also show how these constructions are computed in practice.

In Section 2.1 we begin with a familiar example, focusing on points which are key to the general theory. A wealth of additional examples can be found in the framework of algebraic theories which we recall in Section 2.2. In Section 2.3 we extend this discussion to the simplicially enriched context. Finally we recall some relevant facts about operads in Section 2.4. In these last two sections we introduce two of our primary classes of examples: Simplicial algebraic theories and operads.

2.1. Monadicity and categories of algebras

Given a set S we can form the free group FS on S whose underlying set consists of all finite reduced words whose letters are signed elements of S. Multiplication is then defined by composing words. We can also take a group G, forget its group structure, and regard it is as a set $X = UG$. These constructions are clearly functorial and participate in an adjunction

$$\mathcal{G}roup \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{S}et$$

where U is right adjoint to F. Let $T = UF$ denote the endofunctor of $\mathcal{S}et$ given by the composite of these two functors.

The unit of this adjunction is a natural transformation $e\colon \mathrm{Id} \to T$ given by taking an element of a set to its associated word of length one. Using the underlying group structure on $X = UG$ one can multiply the elements in a word to obtain a *structure map*

$$\mu_X\colon TX \to X.$$

Alternatively we could construct this map by applying U to the counit

$$\varepsilon\colon FU \to \mathrm{Id}$$

of this adjunction. In particular, we have such a map for anything in the image of T and obtain a natural transformation

$$\mu_T\colon T^2 \to T.$$

The (large) category of endofunctors of $\mathcal{S}et$ admits a monoidal structure under composition and we can see that (T, e, μ_T) is an associative monoid in this category, in other words, T is a *monad* on $\mathcal{S}et$.

In the case of $X = UG$ we see that the map μ_X is compatible with this structure in the sense that the two double composites of straight arrows in (2.1) are equal and each composite of a curved arrow followed by a straight arrow is the identity morphism.

(2.1)
$$\begin{tikzcd} TTX \arrow[r, shift left, "\mu_T"] \arrow[r, shift right, "T\mu_X"'] & TX \arrow[l, bend right, "e_{TX}"'] \arrow[r, "\mu_X"] & X \arrow[l, bend right, "e_X"'] \end{tikzcd}$$

An object $X \in \mathcal{S}et$ with a map $\mu_X\colon TX \to X$ satisfying these identities is called a T-algebra in $\mathcal{S}et$. We obtain a category $\mathcal{S}et_T$ of T-algebras in $\mathcal{S}et$ by restricting to those set maps which commute with the structure morphisms. To be explicit, the morphisms between two T-algebras (X, μ_X) and (Y, μ_Y) are those maps $f\colon X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "\mu_X"'] & TY \arrow[d, "\mu_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

or alternatively:

(2.2)
$$\mathcal{S}et_T(X,Y) = \mathrm{eq}\Big[\mathcal{S}et(X,Y) \overset{(\mu_Y)_* \circ T}{\underset{\mu_X^*}{\rightrightarrows}} \mathcal{S}et(TX,Y)\Big].$$

The category of T-algebras in $\mathcal{S}et$ admits an obvious forgetful functor to $\mathcal{S}et$ and we saw above that the forgetful functor $U\colon \mathcal{G}roup \to \mathcal{S}et$ factors through $\mathcal{S}et_T$. It is not difficult to see that the latter functor defines an equivalence of categories $\mathcal{G}roup \simeq \mathcal{S}et_T$. Indeed, if G is a group then we can see that some of the maps in (2.1) can be realized by applying U to following diagram of groups:

(2.3)
$$\begin{tikzcd} FTUG \arrow[r, shift left, "\varepsilon FU"] \arrow[r, shift right, "F\mu_{UG}"'] & FUG \arrow[l, bend right, "e"'] \arrow[r] & G. \end{tikzcd}$$

The map on the right exhibits G as the coequalizer of the two straight arrows on the left. Moreover, the map e exhibits this coequalizer as a reflexive coequalizer. In this sense we see that every group has a *functorial resolution* by free groups. The forgetful functor from $\mathcal{S}et_T$ to $\mathcal{S}et$ admits a left adjoint F_T which factors T as $T = UF_T$. Similarly, we see that every T-algebra admits a functorial resolution by free T-algebras. After forgetting down to $\mathcal{S}et$ these coequalizer diagrams become split coequalizer diagrams [15, Lem. 4.3.3], i.e., diagrams of the form (2.1). Split coequalizer diagrams have the useful property that they are preserved by *all* functors [14, Prop. 2.10.2].

Using these functorial resolutions and that a morphism of groups is an isomorphism if and only if it induces an isomorphism between the underlying sets, we can see that the lifted functor $U\colon \mathcal{G}roup \to \mathcal{S}et_T$ is essentially surjective. By applying the functorial resolution again and (2.2) one can now see that this functor is full and faithful and $\mathcal{G}roup \simeq \mathcal{S}et_T$.

These arguments are completely general:

Theorem 2.4 (Barr-Beck/Monadicity)**.** Any functor $U\colon \mathscr{D} \to \mathscr{C}$ which admits a left adjoint F lifts to a functor to the category of $T = UF$-algebras in $\mathscr{C}$. Moreover this functor is an equivalence of categories if and only if

a. U is conservative, i.e., a map f in $\mathscr{D}$ is an isomorphism if and only if Uf is.
b. For every T-algebra G, if U takes a pair of arrows of the form (2.3) to a split coequalizer, then the pair of arrows in (2.3) admits a coequalizer which is preserved by U.

Proof. This version of the Barr-Beck theorem is a slight variation of [15, Thm. 4.4.4]. Here we assume the existence of a left adjoint, which does not appear there, and our condition (b) is slightly weaker than what is assumed there. But Borceux's argument applies without change. □

Theorem 2.4 can be used to identify many categories as categories of algebras over a monad. Since we want $\mathscr{C}_T$ to have an ample supply of colimits and limits for constructing model structures we postpone introducing these examples for a moment so that we can record when such constructions exist.

Proposition 2.5. [15, Props. 4.3.1, 4.3.2] Suppose T is a monad acting on $\mathscr{C}$, then

1. The forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ creates all limits.
2. The forgetful functor $U\colon \mathscr{C}_T \to \mathscr{C}$ creates all colimits which commute with T in $\mathscr{C}$.

Proposition 2.6. [18, Prop. II.7.4] Suppose $\mathscr{C}$ is cocomplete and T commutes with reflexive coequalizers, then $\mathscr{C}_T$ is cocomplete and the forgetful functor creates all reflexive coequalizers.

Alternatively, if we suppose that $\mathscr{C}$ is bicomplete and T preserves κ-filtered colimits for some regular cardinal κ then $\mathscr{C}_T$ is bicomplete by [15, Prop. 4.3.6]. We often want T, or equivalently U, to preserve *both* filtered colimits and reflexive coequalizers (for some examples where this does not hold see [15, § 4.6]). In such a case we can apply the following useful form of the Barr-Beck theorem provided we restrict to *locally presentable categories* [15, § 5.2].

Proposition 2.7. Suppose $U\colon \mathscr{D} \to \mathscr{C}$ is a conservative functor between two locally presentable categories such that

a. U preserves limits,
b. U creates κ-filtered colimits for some regular cardinal κ,
c. and U creates reflexive coequalizers.

Then U admits a left adjoint F, $\mathscr{D}$ is equivalent to the category of $T = UF$-algebras in $\mathscr{C}$, and T commutes with reflexive coequalizers and κ-filtered colimits.

The above results illustrate the importance of reflexive coequalizers and filtered colimits in $\mathscr{C}_T$. These are particular examples of *sifted* colimits, which are colimits indexed over $\mathscr{I}$ such that the diagonal map $\mathscr{I} \to \mathscr{I} \times \mathscr{I}$ is final. Sifted colimits can also be characterized as those colimits which commute with finite products in $\mathcal{S}et$. One of the main results of [1, Thm. 2.1] is that if $\mathscr{C}$ is finitely cocomplete then T commutes with all (κ-)sifted colimits if and only if T commutes with all reflexive coequalizers and (κ-)filtered colimits.

2.2. Algebraic theories

Monads which commute with sifted colimits arise naturally in the study of algebraic theories in the sense of Lawvere [32]. Recall that an (algebraic) theory is a category $\mathscr{T}$ equipped with a product preserving functor $i\colon \mathcal{F}in\mathcal{S}et^{\mathrm{op}} \to \mathscr{T}$ which is essentially surjective. If we let $\underline{n} \in \mathcal{F}in\mathcal{S}et^{\mathrm{op}}$ be a set with n elements, then, since we are working in the opposite category, $\underline{n} \cong \underline{1}^{\times n}$. So $\mathscr{T}$ is equivalent to a category whose objects are $\{i(\underline{1})^n\}_{n\in\mathbb{N}}$. If $\mathscr{C}$ is a category with finite products, a $\mathscr{T}$-model in $\mathscr{C}$ is a product preserving functor $A\colon \mathscr{T} \to \mathscr{C}$. The collection of $\mathscr{T}$-models in $\mathscr{C}$ forms a category $\mathscr{T}$-$\mathscr{C}$ whose morphisms are natural transformations.

We should think of $\mathscr{T}$ as encoding the operations on an object of $\mathscr{T}$-$\mathscr{C}$. For example, suppose k is a commutative ring and define a theory $\mathscr{T}$ as the subcategory of the opposite category of k-algebras whose nth object $i(\underline{n})$ is the free k-algebra $k\langle x_1,\cdots,x_n\rangle$.

Note that for each k-algebra A, we obtain a $\mathscr{T}$-model in $\mathcal{S}et$ by

$$i(\underline{n}) \mapsto k\text{-}\mathcal{A}lg(k\langle x_1,\cdots,x_n\rangle, A) \cong A^n.$$

Conversely, if $A \in \mathscr{T}$-$\mathcal{S}et$ we can identify A with the set $A(i(\underline{1}))$ equipped with the operations encoded by the functor A. For example, consider the maps in

$$\mathscr{T}(i(\underline{2}), i(\underline{1})) \cong k\text{-}\mathcal{A}lg(k\langle x_1\rangle, k\langle x_1,x_2\rangle)$$

which send x_1 to $x_1 + x_2$ and $x_1 \cdot x_2$ respectively. These two maps define natural operations

$$(-)+(-)\colon A(i(\underline{1}))^2 \to A(i(\underline{1}))$$
$$(-)\cdot(-)\colon A(i(\underline{1}))^2 \to A(i(\underline{1})).$$

The first map is commutative since $x_1 + x_2 = x_2 + x_1$, while the latter generally is not. By combining maps in $\mathscr{T}$ we can see that the latter operation will distribute over the former. All of these operations and their relations coming from $\mathscr{T}$ show that $A(i(\underline{1}))$ is a k-algebra.

Example 2.8.

1. Let $\mathscr{T}_{\mathrm{Gp}}$ be the category whose objects are indexed by natural numbers and whose morphisms are
$$\mathscr{T}_{\mathrm{Gp}}(m,n) = \mathcal{G}roup(F\{n\}, F\{m\}),$$
where $F\{m\}$ is the free group on m elements, then $\mathscr{T}_{\mathrm{Gp}}$-$\mathcal{S}et$ is equivalent to the category of groups.

2. Let G be a group and let $\mathcal{T}_G$ be the theory defined as in (1) but with

$$\mathcal{T}_G(m,n) = G\text{-}\mathcal{S}et(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free G-set on m elements, then $\mathcal{T}_G$-$\mathcal{S}et$ is equivalent to the category of G-sets.
3. Let k be a commutative ring and $\mathcal{T}_{Lie_k}$ be the theory defined as in (1) but with

$$\mathcal{T}_{Lie_k}(m,n) = \mathcal{L}ie_k(F\{n\}, F\{m\}),$$

where $F\{m\}$ is the free Lie algebra over k on m elements, then $\mathcal{T}_{Lie_k}$-$\mathcal{S}et$ is equivalent to the category of Lie algebras over k.
4. Let $\mathcal{T}_{C^\infty}$ be the theory defined as in (1) but with $\mathcal{T}_{C^\infty}(m,n) = \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{R}^n)$, the set of smooth maps from $\mathbb{R}^m$ to $\mathbb{R}^n$, then $\mathcal{T}_{C^\infty}$-$\mathcal{S}et$ is equivalent to the category of C^∞-rings [17, 38].

The list in Example 2.8 is far from comprehensive and is limited only by the authors' imagination and the readers' patience.

If $\mathcal{T}$ is a theory, we obtain $\mathcal{T}$-models $\mathcal{T}\{\underline{m}\}$ in $\mathcal{S}et$ by setting $\mathcal{T}\{\underline{m}\}(-) = \mathcal{T}(i(\underline{m}), -)$, which we can think of as the free objects on a set of m-elements. This construction lifts to a (covariant!) functor $\mathcal{T}\{-\}\colon \mathcal{F}in\mathcal{S}et \to \mathcal{T}$-$\mathcal{S}et$. Since $\mathcal{S}et$ is the closure of $\mathcal{F}in\mathcal{S}et$ under sifted colimits, and sifted colimits commute with products in $\mathcal{S}et$ we see that we can canonically prolong this to a functor from $\mathcal{S}et$. This functor admits a forgetful right adjoint given by evaluating at $\underline{1}$.

Just as in Section 2.1, we can compose these adjoints to obtain a monad $T = U\mathcal{T}\{-\}$ on $\mathcal{S}et$. More explicitly, the formula for the left Kan extension shows

$$TX = \int^{\underline{n} \in \mathcal{F}in\mathcal{S}et^{\mathrm{op}}} \mathcal{T}(i(\underline{n}), i(\underline{1})) \times X^n. \tag{2.9}$$

Since $\mathcal{T}$-$\mathcal{S}et$ is locally presentable with a conservative right adjoint U which creates sifted colimits we can apply Proposition 2.7 and see that $\mathcal{T}$-$\mathcal{S}et$ is equivalent to the category of $T = U\mathcal{T}\{-\}$ algebras in $\mathcal{S}et$.

Given a T-algebra $X \in \mathcal{T}$-$\mathcal{S}et$ we can consider the category $(\mathcal{T}\text{-}\mathcal{S}et)_{\downarrow X}$ of algebras over X. Generally this can not be realized as the category of $\mathcal{S}et$-valued models for an algebraic theory. However, it can be realized as the category of models of a *graded* theory $\mathcal{T} \downarrow X$ (see Example 2.11). This additional generality will prove useful in the identification of the E_2 term of the T-algebra spectral sequence with Quillen cohomology groups in Theorem B (see Section 4.3).

Definition 2.10. For a set S of gradings, an *S-graded theory* $\mathcal{T}$ is a category $\mathcal{T}$ equipped with a product preserving functor $i\colon (\mathcal{F}in\mathcal{S}et^S)^{\mathrm{op}} \to \mathcal{T}$ which is essentially surjective.

When working with graded theories, it is sometimes useful to use the isomorphism of categories

$$\phi\colon \mathcal{F}in\mathcal{S}et^S \cong \mathcal{F}_S$$

where $\mathcal{F}_S$ is the category whose objects are pairs (X,f) where $f: X \to S$ is a map of sets with finite fibers. Morphisms in $\mathcal{F}_S$ are given by commuting triangles over S. For $x = (x_s)_{s\in S} \in \mathcal{F}in\mathcal{S}et^S$, $\phi(x) = (\coprod_s x_s, \coprod_s f_s)$ where each f_s is the unique map of sets $x_s \to \{s\}$. Likewise, for $(X,f) \in \mathcal{F}_S$, $\phi^{-1}(X,f) = (f^{-1}(s))_{s\in S}$.

Here are two prototypical examples of graded theories.

Example 2.11.

1. Fix an abelian group M. The category of abelian groups over M is the category of $(\mathcal{T}_{\mathrm{Ab}} \downarrow M)$-models in $\mathcal{S}et$ where $\mathcal{T}_{\mathrm{Ab}} \downarrow M$ is a theory graded on the underlying set of M and is defined as follows: The objects of $\mathcal{T}_{\mathrm{Ab}} \downarrow M$ are the objects of $\mathcal{F}_M^{\mathrm{op}}$. For each $x = (x_m)_{m\in M}$, let $i(x) = \phi(x) = (X, f)$ be the corresponding object of $\mathcal{F}_M^{\mathrm{op}}$ and let $F_{\mathrm{Ab}}X$ be the free abelian group over M on the elements of X, with structure map $F_{\mathrm{Ab}}X \to M$ determined by the set map $f: X \to M$. If $i(x) = (X, f)$ and $i(x') = (X', f)$ are two objects of $\mathcal{T}_{\mathrm{Ab}} \downarrow M$, then we define

$$(\mathcal{T}_{\mathrm{Ab}} \downarrow M)((X,f),(X',f')) = \mathcal{A}b\mathcal{G}roup_{\downarrow M}\left(F_{\mathrm{Ab}}X', F_{\mathrm{Ab}}X\right).$$

2. The category of $\mathbb{Z}$-graded abelian groups is the category of $\mathcal{T}_{\mathrm{Ab}}^{\mathbb{Z}}$-models in $\mathcal{S}et$ where $\mathcal{T}_{\mathrm{Ab}}^{\mathbb{Z}}$ is the $\mathbb{Z}$-graded theory defined as follows: For $j \in \mathbb{Z}$ let $\mathbb{Z}^t[j]$ denote the free abelian group on t elements concentrated in degree j. If $i(t)$ and $i(t')$ are two elements of $\mathcal{T}_{\mathrm{Ab}}^{\mathbb{Z}}$ with $t = (\underline{t}_j)_{j\in\mathbb{Z}}, t' = (\underline{t'}_k)_{k\in\mathbb{Z}} \in (\mathcal{F}in\mathcal{S}et^{\mathbb{Z}})^{\mathrm{op}}$ then we define

$$\mathcal{T}_{\mathrm{Ab}}^{\mathbb{Z}}(i(t), i(t')) = \mathcal{A}b\mathcal{G}roup^{\mathbb{Z}}\left(\bigoplus_{k\in\mathbb{Z}} \mathbb{Z}^{t'_k}[k], \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}^{t_j}[j]\right) \cong \prod_{k\in\mathbb{Z}} \mathcal{A}b\mathcal{G}roup\left(\mathbb{Z}^{t'_k}[k], \mathbb{Z}^{t_k}[k]\right).$$

The theories in Example 2.8 can all be extended to the graded case similarly and in general overcategories coming from graded theories are graded theories.

Remark 2.12. The evaluation operation on the category of models in $\mathcal{S}et$ of an S-graded theory $\mathcal{T}$ defines a forgetful functor to $\mathcal{S}et^S \cong \prod_S \mathcal{S}et$. This functor is finitary (meaning it preserves countable filtered colimits) and monadic (meaning it satisfies Theorem 2.4) with associated monad T, so the category of $\mathcal{T}$-models in $\mathcal{S}et$ is equivalent to the category of T-algebras in $\mathcal{S}et^S$.

This construction is part of a correspondence demonstrated in [2, App. A] between categories of $\mathcal{S}et$-valued models over S-graded theories and finitary monadic categories over $\mathcal{S}et^S$. Their results can in turn be used to show a correspondence between the latter and algebraic categories in the sense of [40, 41].

The following result will help us in Proposition 3.5 connect the machinery of algebraic theories to Quillen cohomology.

Proposition 2.13. Let $\mathcal{T}$ be a graded theory and X a $\mathcal{T}$-model in $\mathcal{S}et$. Let $\mathcal{T} \downarrow X$ be the S-graded theory whose category of models is $(\mathcal{T}\text{-}\mathcal{S}et)_{\downarrow X}$, the category of objects over X in $\mathcal{T}$-$\mathcal{S}et$. Then there is an S-graded theory $(\mathcal{T} \downarrow X)_{\mathrm{ab}}$ such that the category of $(\mathcal{T} \downarrow X)_{\mathrm{ab}}$-models in $\mathcal{S}et$, $(\mathcal{T} \downarrow X)_{\mathrm{ab}}$-$\mathcal{S}et$, is equivalent to the category of abelian group objects in $(\mathcal{T} \downarrow X)$-$\mathcal{S}et$.

These two categories are monadic over $\mathcal{S}et^S$ with associated monads $(T\downarrow X)_{\mathrm{ab}}$ and $T\downarrow X$. The forgetful functor

$$\mathcal{S}et^S_{(T\downarrow X)_{\mathrm{ab}}} \to \mathcal{S}et^S_{T\downarrow X}$$

is monadic with left adjoint Ab.

Proof. We have already noted that the category of models over X is a graded theory. Let S denote the grading for this theory. As a consequence of [15, Thm. 3.11.3] the category of abelian group objects in the category of models for a theory is a category of models for a new theory. As noted in [12, § 3.3] this argument passes to the S-graded case, mutatis mutandis, to yield an S-graded theory for the abelian group objects.

These are both locally presentable categories. Since the forgetful functor is conservative and limits and sifted colimits in these categories are both calculated in $\mathcal{S}et^S$ we can apply Proposition 2.7 to complete the proof. □

2.3. Simplicial categories of T-algebras

The theory of T-algebra limits and colimits from Section 2.2 admits a straightforward extension to the enriched context. Since we are interested in studying the *space* of maps between two T-algebras, modeled as a simplicial set, we give this extension in the case that $\mathscr{C}$ is a simplicial category.[1] To obtain categorical information analogous to the previous section we will replace all of our categories with simplicial categories, all of our functors with simplicial functors, and all of our natural transformations with simplicial natural transformations. For general background on enriched categories and functors between them the reader is encouraged to consult [15, § 6.2] or [29].

Recall that we require a simplicial category $\mathscr{C}$ to have a tensor bifunctor

$$\otimes\colon s\mathcal{S}et \times \mathscr{C} \to \mathscr{C}.$$

This is related to the simplicial mapping functor $\mathscr{C}(-,-)$ and the simplicial cotensor $(-)^{-}$ via natural adjunction isomorphisms

$$s\mathcal{S}et(K, \mathscr{C}(C,D)) \cong \mathscr{C}(K \otimes C, D) \cong \mathscr{C}(C, D^K).$$

Proposition 2.14. Suppose that

a. $\mathscr{C}$ is a bicomplete simplicial category.
b. T is a simplicial monad acting on $\mathscr{C}$.
c. T commutes with either
   (i) reflexive coequalizers or
   (ii) filtered colimits.

Then $\mathscr{C}_T$ is a bicomplete simplicial category such that

1. The forgetful functor $\mathscr{C}_T \to \mathscr{C}$ creates limits and cotensors.
2. The simplicial tensor is constructed as follows:

$$K \otimes_T X = \operatorname{coeq}\left[\begin{tikzcd} F_T(K \otimes TUX) \arrow[r, shift left, "F_T(K\otimes\mu)"] \arrow[r, shift right, "\alpha"'] & F_T(K \otimes UX) \arrow[l, bend right=40, "F_T(K\otimes e)"'] \end{tikzcd} \right]. \tag{2.15}$$

Here $\alpha\colon F_T(K \otimes TUX) \to F_T(K \otimes UX)$ is adjoint to the assembly map $K \otimes TUX \to T(K \otimes UX)$.

Proof. First we check that $\mathscr{C}_T$ is bicomplete: By Proposition 2.5 $\mathscr{C}_T$ is complete and U creates limits. Under hypothesis (c.i) we can apply Proposition 2.6 to see that $\mathscr{C}_T$ is cocomplete. When hypothesis (c.ii) holds, cocompleteness follows from [15, Prop. 4.3.6].

The hom spaces of $\mathscr{C}_T$ are defined by taking the equalizer, in $s\mathcal{S}et$, of the the obvious analogue of (2.2). The fact that U creates cotensors appears in [18, Prop. VII.2.10]. In order for the adjunctions to hold the tensor must be defined by (2.15). □

[1] Although we will normally distinguish between topological spaces and simplicial sets, we will refer to both as spaces in the case of mapping objects. We justify this abuse by noting that we are primarily interested in *derived* mapping spaces, which are only homotopy types, so it is not necessary to distinguish between the choice of model.

Note that under the hypotheses of Proposition 2.14, if T commutes with reflexive coequalizers we can compute (2.15) in $\mathscr{C}$ by Proposition 2.5.

Graded algebraic theories are extended similarly to the simplicial context: Regarding the category of finite sets as a simplicially enriched category with discrete mapping objects, a simplicial algebraic theory is just a product preserving functor $(\mathcal{F}in\mathcal{S}et^S)^{\mathrm{op}} \to \mathscr{T}$ to a simplicially enriched category $\mathscr{T}$ which is essentially surjective. Similarly, a T-model in a simplicially enriched category $\mathscr{C}$ with finite products is just a product preserving simplicial functor $\mathscr{T} \to \mathscr{C}$.

Example 2.16. Each of the examples listed in Example 2.8 and their graded counterparts naturally defines a simplicial theory. The $\mathscr{T}$-models in simplicial sets are equivalent to their simplicial analogues.

Proposition 2.17. Let T be the simplicial monad acting on $s\mathcal{S}et^S$ associated to an S-graded simplicial algebraic theory $\mathscr{T}$. Then the category $s\mathcal{S}et^S_T$ of T-algebras is a bicomplete simplicial category with tensor defined by (2.15).

If $S = *$ and $\mathscr{T}$ is an ordinary theory regarded as a constant simplicial theory then for each $K \in s\mathcal{S}et$ and $X \in s\mathcal{S}et_T$ we have the identification

$$(K \otimes X)_n = \coprod_{k \in K_n}^{\mathscr{T}\text{-}\mathcal{S}et} X_n.$$

2.4. Monads from operads

A *symmetric sequence* in $s\mathcal{S}et$ is a sequence

$$C = \{C(n)\}_{n \geq 0}$$

where $C(n)$ is a simplicial set with a right Σ_n-action. A map of symmetric sequences is a levelwise equivariant map.

For the remainder of this section we assume that $\mathscr{C}$ is simplicial symmetric monoidal category with tensor $\otimes$ such that:

a. $\otimes$ distributes over countable coproducts in $\mathscr{C}$ and
b. there is a symmetric monoidal functor $i : s\mathcal{S}et \to \mathscr{C}$, such that the tensor of a space K and an object X of $\mathscr{C}$ is defined by $iK \otimes X$.

Now given a symmetric sequence C, we have an associated functor $T_C : \mathscr{C} \to \mathscr{C}$ defined on objects by

$$T_C(X) = \coprod_{n \geq 0} C(n) \otimes_{\Sigma_n} X^{\otimes n} \cong \int^{\underline{n} \in \mathrm{Iso}\mathcal{S}et} C(n) \otimes X^{\otimes n}. \tag{2.18}$$

A map of symmetric sequences yields a natural transformation of functors, and this construction yields a functor from symmetric sequences to endofunctors of $\mathscr{C}$.

There is a symmetric monoidal product on symmetric sequences, which we will also denote by $\otimes$:

$$(C \otimes D)(n) = \coprod_{i+j=n} C(i) \times D(j) \times_{\Sigma_i \times \Sigma_j} \Sigma_n$$

Since the symmetric monoidal structure on $\mathscr{C}$ distributes over coproducts we see:

$$T_C \otimes T_D \cong T_{C \otimes D}.$$

Now we define the circle product by:

$$(C \circ D)(n) = \left(\coprod_{i \geq 0} C(i) \times_{\Sigma_i} D^{\otimes i} \right)(n).$$

This is part of a monoidal structure on symmetric sequences such that the construction $C \mapsto T_C$ defines a monoidal functor to the category of endofunctors with composition product. An operad $\mathscr{O}$ is a symmetric sequence which is a monoid for the circle product; the associated endofunctor is then a monad (see [44, § 11] for additional details).

Remark 2.19. The category of operads in $s\mathcal{S}et$ can be constructed as the category of $s\mathcal{S}et$-valued models for a graded simplicial algebraic theory. As in [43, App. A] one can construct the free monoid with respect to the circle product on a symmetric sequence. Regarding the Σ_n-set $\Sigma_n \times \underline{i}$ as a symmetric sequence concentrated in degree n (and the empty set elsewhere), we can apply this free construction to the symmetric sequences $\{\Sigma_n \times \underline{i}\}_{(n,i) \in \mathbb{N} \times \mathbb{N}}$ to define an $\mathbb{N}$-graded algebraic theory whose algebras are operads in $\mathcal{S}et$. The category of simplicial operads is the associated category of models in $s\mathcal{S}et$.

The following standard result gives criteria for identifying when the category of algebras over an operad is simplicially enriched.

Proposition 2.20. Suppose that $\mathscr{C}$ is a bicomplete simplicial symmetric monoidal category such that:

a. There is a symmetric monoidal functor $i\colon s\mathcal{S}et \to \mathscr{C}$ defining the simplicial tensor.
b. The monoidal product in $\mathscr{C}$ commutes with countable coproducts and either
   (i) reflexive coequalizers or
   (ii) filtered colimits.

Then for any operad $\mathscr{O}$ of simplicial sets, the category of $\mathscr{O}$-algebras in $\mathscr{C}$ is a bicomplete simplicial category.

The hypotheses concerning colimits for this proposition hold whenever the symmetric monoidal structure comes from a *closed* symmetric monoidal structure and hence distributes over all colimits. For example, simplicial sets, simplicial abelian groups, and simplicial R-modules all satisfy the conditions of Proposition 2.20 with their respective closed symmetric monoidal structures. The categories of pointed compactly generated weak Hausdorff spaces or pointed simplicial sets, each equipped with the smash product, satisfy these conditions. Any of the standard closed symmetric monoidal categories of spectra also satisfy the hypotheses.

3. Homotopy theory of T-algebras

In Section 3.1 we recall conditions that guarantee that the category of T-algebras has a suitable homotopy theory. After establishing the existence of a model structure, we construct functorial simplicial resolutions of algebras in Section 3.2 which are used in the construction of the T-algebra spectral sequence.

Here, we choose to work in the context of simplicial model categories. A disadvantage of this approach is that some of our assumptions—most notably the existence of colimits/limits and the standard issues concerning cofibrancy and fibrancy—should not be strictly necessary (see for example

[34, § 6.2]). An advantage of this approach is that the theory is well-developed, well-understood, and relatively straightforward to apply to many categories of interest.

We have gathered the relevant results from the literature in the interest of having a single reference for determining whether a category of T-algebras admits a simplicial model structure. The background material for this section can be found in [40, 25, 23] or the appendices of [33].

3.1. Model structures on T-algebras

Let $\mathscr{C}$ be a simplicial model category and T a simplicial monad acting on $\mathscr{C}$. We will now recall conditions which guarantee the simplicial structure on $\mathscr{C}_T$ is part of a simplicial model structure [40]. Such model categories $\mathscr{M}$ satisfy the following two equivalent forms of Quillen's corner axiom.

SM7: Given any cofibration $f \in \mathcal{sSet}(K,L)$ and fibration $g \in \mathscr{M}(A,B)$, the induced morphism

$$A^L \longrightarrow A^K \times_{B^K} B^L$$

is a fibration, which is a weak equivalence if either f or g is.

SM7a: Given cofibrations $f \in \mathcal{sSet}(K,L)$ and $g \in \mathscr{M}(A,B)$, the induced morphism

$$K \otimes B \coprod_{K \otimes A} L \otimes A \longrightarrow L \otimes B$$

is a cofibration, which is a weak equivalence if either f or g is.

Definition 3.1. Let $\mathscr{C}$ be a (simplicial) model category. A (simplicial) monad T acting on $\mathscr{C}$ is *(simplicial) Quillen* if

a. $\mathscr{C}_T$ has a (simplicial) model structure such that the forgetful functor $U : \mathscr{C}_T \to \mathscr{C}$ is a (simplicial) right Quillen functor.
b. A map f of T-algebras is a weak equivalence if and only if Uf is a weak equivalence.

A convenient way to show that T is (simplicial) Quillen is to induce a (simplicial) model structure on $\mathscr{C}_T$ via F_T. We can do this if $\mathscr{C}$ is a cofibrantly generated (simplicial) model category and T satisfies some mild hypotheses. In this case $\mathscr{C}$ has sets of generating cofibrations I and acyclic cofibrations J which are used to detect acyclic fibrations and fibrations respectively. These sets of maps satisfy smallness hypotheses which are used to apply Quillen's small object argument and prove the lifting axioms.

Suppose that $\mathscr{C}$ is a model category and a functor

$$U : \mathscr{D} \to \mathscr{C}$$

admits a left adjoint. Then we say that U *right induces* a model structure on $\mathscr{D}$ if $\mathscr{D}$ admits a model structure such that a map f is a fibration (resp. weak equivalence) if and only if Uf is a fibration (resp. weak equivalence).

Theorem 3.2 (Cf. [48, App. A Thm. 1.4])**.** Suppose that $\mathscr{C}$ is a cofibrantly generated simplicial model category with generating (acyclic) cofibrations I (resp. J) and $T = UF_T$ is a simplicial monad acting on $\mathscr{C}$ satisfying Proposition 2.14.

If the domains of $F_T I$ (resp. $F_T J$) are small relative to $F_T I$-cells (resp. $F_T J$-cells) and applying U to any $F_T J$-cell complex yields a weak equivalence in $\mathscr{C}$ then U right induces a cofibrantly generated simplicial model category structure on $\mathscr{C}_T$.

Remark 3.3. In practice, checking the smallness conditions is relatively easy. In fact, it is automatic when the underlying categories are locally presentable. So most of the work required to apply Theorem 3.2 involves checking that applying U to an $F_T J$-cell complex yields a weak equivalence. This can be verified (see [46, Lem. B2]) by showing the following two properties are satisfied.

1. There is a 'fibrant replacement' functor $Q : \mathscr{C}_T \to \mathscr{C}_T$ and a natural transformation $\mathrm{Id} \to Q$ such that for all $X \in \mathscr{C}_T$, the natural map $UX \to UQX$ is a fibrant replacement.
2. If UX is fibrant then applying U to the canonical factorization $X \to X^{\Delta^1} \to X^{\partial\Delta^1} \cong X \times X$ of the diagonal yields a weak equivalence followed by a fibration.

In our setup, the second property follows from the fact that U preserves cotensors and $\mathscr{C}$ is a simplicial model category. To verify the first property one can sometimes show that the fibrant replacement functor $\mathscr{C}$ lifts to an endomorphism of $\mathscr{C}_T$. This is automatic if every object is fibrant in $\mathscr{C}$. Since the two fibrant replacement functors Ex^∞ and $\mathrm{Sing}_*|-|$ on simplicial sets are product preserving we can use either of them as fibrant replacement functors for the category of simplicial T-algebras associated to a (graded) theory.

Proposition 3.4 (Cf. [47, Thm. 3.1])**.** If T is the monad associated to a (graded) algebraic theory on simplicial sets (such as, e.g., those in Example 2.16 or Remark 2.19), then T is simplicial Quillen.

Proposition 3.5. Let $\mathscr{T}$ be a graded theory and X a model in $s\mathcal{S}et$ for this theory. Let B be the category of $\mathscr{T}$-models over X in $s\mathcal{S}et$. Then there is an S-graded theory whose models in $s\mathcal{S}et$, which we will denote by A, is equivalent to the category of abelian group objects in B.

These two categories are monadic over $s\mathcal{S}et^S$ with associated monads T_A and T_B and the forgetful functor

$$\iota\colon s\mathcal{S}et^S_{T_A} \to s\mathcal{S}et^S_{T_B}$$

is monadic with left adjoint Ab. Its associated monad is simplicial Quillen.

Proof. The argument from Proposition 2.13 proves the claims about the underlying categories. We just need to check that ι is a right Quillen functor (it is obviously simplicial). In both categories a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) in $s\mathcal{S}et^S$. Since the forgetful functor $s\mathcal{S}et^S_{T_A}$ to $s\mathcal{S}et^S$ factors through ι the result follows. □

Definition 3.6. Given a simplicial model category $\mathscr{C}$ we define the derived mapping space

$$\mathscr{C}^d(X,Y) = \mathscr{C}(X^c, Y^f)$$

where X^c is a cofibrant replacment of X and Y^f is a fibrant replacement of Y. We now define

$$ho\mathscr{C}(X,Y) = \pi_0 \mathscr{C}^d(X,Y).$$

It follows easily from Axiom [SM7] that the derived mapping space is well-defined up to weak equivalence and therefore $ho\mathscr{C}(X,Y)$ is well-defined. That this definition agrees with other constructions of the set of homotopy classes of morphisms, and that these hom-sets assemble into a homotopy category $ho\mathscr{C}$ can be found in [21, Prop. II.3.10 + § II.3].

Proposition 3.7. Suppose that the forgetful functor $U : \mathscr{C}_T \to \mathscr{C}$ is a simplicial Quillen right adjoint. Then the monad T induces a monad hT on $ho\mathscr{C}$ such that the forgetful functor $ho(\mathscr{C}_T) \to ho\mathscr{C}$ factors through $(ho\mathscr{C})_{hT}$.

Proof. Quillen adjoints induce adjoints between the homotopy categories and consequently a monad action on $ho\mathscr{C}$ given by the composite. The right adjoint between the homotopy categories always lands in the category of algebras over this monad. □

Definition 3.8. Let $\mathscr{T}$ be an S-graded theory and X a $\mathscr{T}$-model in $s\mathcal{S}et$. Let

$$\iota\colon s\mathcal{S}et^S_{(T\downarrow X)_{\mathrm{ab}}} \to s\mathcal{S}et^S_{T\downarrow X}$$

be the Quillen right adjoint from Proposition 3.5.

If M is in $\mathcal{S}et^S_{(T\downarrow X)_{\mathrm{ab}}}$ and $Y \in s\mathcal{S}et^S_{T\downarrow X}$, then the sth *Quillen cohomology* of Y with coefficients in M is defined to be the group

$$H^s_{Q,X}(Y;M) := hos\mathcal{S}et^S_{T\downarrow X}(Y,\iota\Sigma^s M)$$

where $\Sigma^s M$ is the sth suspension of M (see [40]).

3.2. Simplicial resolutions of T-algebras

If T is a monad acting on $\mathscr{C}$, then applying T levelwise to simplicial objects in $\mathscr{C}$ yields a monad acting on the category $s\mathscr{C}$ of simplicial objects which we will also denote by T. Now to construct a spectral sequence computing the homotopy groups of the space $\mathscr{C}_T(X,Y)$ we would like to resolve X, meaning that we want to replace X by a nice simplicial T-algebra $X_\bullet$ such that $\mathscr{C}_T(|X_\bullet|,Y) \simeq \mathscr{C}^d_T(X,Y)$. If T is a monad acting on $\mathscr{C}$, then applying T levelwise to simplicial objects in $\mathscr{C}$ yields a monad, also denoted by T, acting on the category of simplicial objects in $\mathscr{C}$.

Definition 3.9. Suppose X is a T-algebra in $\mathscr{C}$. The *bar resolution* (also called the cotriple resolution) of X is the simplicial T-algebra

$$B_\bullet X = B_\bullet(F_T,T,UX) = B_\bullet(F_TU,F_TU,X)$$

with $B_nX = (F_TU)^{n+1}X$ and face and degeneracy maps induced from the monad structure on $T = UF_T$ and the T-algebra structure on X.

Note that the counit $F_TUX \to X$ extends to a map of simplicial T-algebras

$$\varepsilon\colon B_\bullet X \to X \tag{3.10}$$

where we regard the target as a constant simplicial object. By applying U to (3.10) we obtain a map $\varepsilon\colon T^{\bullet+1}UX \to UX$ in $s\mathscr{C}$. We also have a simplicial map $e\colon UX \to T^{\bullet+1}UX$ by iterating the unit map $UX \to TUX$.

For a simplicial T-algebra X, there are two relevant geometric realizations. One is realization in the category of T-algebras, and another is realization in the underlying category. We would like to have conditions under which these two notions coincide, i.e., under which U commutes with geometric realizations.

One such condition appears in [18, Prop. X.1.3.v]: If T is given by a coend formula, then U preserves geometric realizations. More precisely, if T is given by a formula such as the one in (2.9), we will show that T commutes with geometric realization and then apply Proposition 3.12 to see that U commutes with geometric realizations.

Proposition 3.11. Let $\mathscr{C}$ be a simplicial category and $F\colon \mathscr{C}\times\mathscr{D}\to\mathscr{C}$ a functor such that for each $d\in\mathscr{D}$, $F(-,d)$ commutes with geometric realizations in $\mathscr{C}$. If T is an endofunctor of $\mathscr{C}$ of the form

$$TX = \int^{d\in\mathscr{D}} G(d)\otimes F(X,d)$$

for some $G\colon \mathscr{D}^{\mathrm{op}}\to s\mathcal{S}et$, then T commutes with geometric realizations.

Proof. Since geometric realization is a coend, the result immediately follows from Fubini's theorem for iterated coends. $\square$

Proposition 3.12. Let $\mathscr{C}$ be a bicomplete simplicial category and T a simplicial monad acting on $\mathscr{C}$. Suppose that $\mathscr{C}_T$ is a bicomplete simplicial category and that T commutes with geometric realization. If $X_\bullet$ is a simplicial object in $\mathscr{C}_T$, then $|UX_\bullet|_{\mathscr{C}}$ is a T-algebra and $U|X_\bullet|_{\mathscr{C}_T}\cong|UX_\bullet|_{\mathscr{C}}$ in $\mathscr{C}_T$. So U commutes with geometric realization if and only if T does.

Proof. Since F_T is a left adjoint it commutes with geometric realization. So if U commutes with geometric realization then so does $T=UF_T$.

Now suppose that T commutes with geometric realization. If we take the geometric realization in the category of T-algebras of a canonical presentation

$$\begin{tikzcd} F_TTUX_\bullet \arrow[r, shift left] \arrow[r, shift right] & F_TUX_\bullet \arrow[l, bend right, "e"'] \arrow[r] & X_\bullet, \end{tikzcd}$$

and then apply U we obtain the following commutative diagram with marked isomorphisms:

$$\begin{tikzcd} U|F_TTUX_\bullet|_{\mathscr{C}_T} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & U|F_TUX_\bullet|_{\mathscr{C}_T} \arrow[l, bend right, "e"'] \arrow[r] \arrow[d, "\cong"] & U|X_\bullet|_{\mathscr{C}_T} \arrow[d, equal] \\ T|TUX_\bullet|_{\mathscr{C}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"'] & T|UX_\bullet|_{\mathscr{C}} \arrow[l, bend right, "e"'] \arrow[r] \arrow[d, "\cong"] & U|X_\bullet|_{\mathscr{C}_T} \arrow[d, "\cong"] \\ |TTUX_\bullet|_{\mathscr{C}} \arrow[r, shift left] \arrow[r, shift right] & |TUX_\bullet|_{\mathscr{C}} \arrow[l, bend right, "e"'] \arrow[r] & |UX_\bullet|_{\mathscr{C}}. \end{tikzcd}$$

The vertical isomorphisms between the first two rows follow from F_T being a left adjoint. The next two vertical isomorphisms on the left follow from our assumption on T and imply the desired lower right hand isomorphism. $\square$

One can interpret the following result as saying that the bar resolution is indeed a resolution.

Proposition 3.13. Suppose $\mathscr{C}$ is a simplicial model category and T is a simplicial Quillen monad acting on $\mathscr{C}$. If T (or equivalently U) commutes with geometric realization, then

$$\varepsilon\colon |B_\bullet X|_{\mathscr{C}_T}\to X$$

is a weak equivalence of T-algebras.

Proof. Because T is Quillen, it suffices to show that $U\varepsilon$ is a weak equivalence in $\mathscr{C}$. This follows from Proposition 3.12 and the following well known lemma. $\square$

Lemma 3.14. [36, Prop. 9.8] Let $X \in \mathscr{C}_T$. The maps e and ε on realization

$$UX \xrightarrow{e} |T^{\bullet+1}UX|_{\mathscr{C}} \xrightarrow{\varepsilon} UX$$

exhibit UX as a strong deformation retract of $|T^{\bullet+1}UX|_{\mathscr{C}}$ in $\mathscr{C}$.

3.3. Reedy cofibrant resolutions of T-algebras

To construct a spectral sequence using the bar resolution of X we require that this resolution is homotopically well behaved, that is we will require it to be a Reedy cofibrant simplicial diagram as described below. If we were to simply take a Reedy cofibrant replacement of the bar resolution we would no longer be able to apply Proposition 3.13 to deduce that the geometric realization of our resolution has the correct homotopy type. To show that a bar resolution is Reedy cofibrant we will apply a useful trick (Proposition 3.17) which makes use of a closely related *almost simplicial* diagram.

Let Δ_0 be the subcategory of Δ with the same objects but whose morphisms are those morphisms of linearly ordered sets which preserve the minimal element. The restriction morphism

$$\iota^* : \mathscr{C}^{\Delta^{\mathrm{op}}} \to \mathscr{C}^{\Delta_0^{\mathrm{op}}}$$

takes a simplicial object and forgets the d_0 face maps (those induced by injections missing the minimal element) while retaining all of the other structure. So one can think of a Δ_0^{op}-shaped diagram as almost a simplicial diagram; it simply lacks the d_0 face maps.

Definition 3.15. Let $X_\bullet$ be in $\mathscr{C}^{\Delta^{\mathrm{op}}}$ (resp. $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$). The nth *latching object* of $X_\bullet$ is

$$L_n(X_\bullet) = \operatorname*{colim}_{[n]\to[k]} X_k,$$

where the colimit is indexed over the non-identity surjections in Δ (this is equal to the set of non-identity surjections in Δ_0).

The category Δ is the prototypical example of a Reedy category [23, § 15.1]. It is immediate from the definition of a Reedy category that the Reedy structure on Δ restricts to Reedy structure on Δ^{op}. This structure is used to make the following:

Definition 3.16. Suppose that $\mathscr{C}$ is a model category. The *Reedy* model structure on $\mathscr{C}^{\Delta^{\mathrm{op}}}$ (resp. $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$) is determined by

a. $f : X_\bullet \to Y_\bullet$ is a (Reedy) weak equivalence if $f_n : X_n \to Y_n$ is a weak equivalence in $\mathscr{C}$ for all $n \geq 0$.
b. $f : X_\bullet \to Y_\bullet$ is a (Reedy) cofibration if the induced map

$$X_n \coprod_{L_n X_\bullet} L_n Y_\bullet \to Y_n$$

is a cofibration in $\mathscr{C}$ for all $n \geq 0$.

To show the bar resolution is Reedy cofibrant in particular cases we will use the following trick:

Proposition 3.17. Suppose $\mathscr{C}$ and $\mathscr{D}$ are model categories and $L\colon \mathscr{D} \to \mathscr{C}$ is a left Quillen functor. Let $X_\bullet$ be a simplicial diagram in $\mathscr{C}$ and $\iota^* X_\bullet \in \mathscr{C}^{\Delta_0^{\mathrm{op}}}$ its restriction. Suppose that there exists a Reedy cofibrant $\widetilde{X}_\bullet \in \mathscr{D}^{\Delta_0^{\mathrm{op}}}$ such that $L\widetilde{X}_\bullet \cong \iota^* X_\bullet$. Then $X_\bullet$ is Reedy cofibrant.

Proof. By definition $X_\bullet$ is Reedy cofibrant if for each non-negative n the latching map

$$L_n X_\bullet \to X_n$$

is a cofibration. Note that the latching object and map depend only on the restriction of $X_\bullet$ to the subcategory $\Delta_{\mathrm{surj}}^{\mathrm{op}}$, where Δ_{surj} consists of all objects $[n]$ but only surjective maps. In particular it suffices to show $\iota^* X_\bullet$ is Reedy cofibrant. Since L is a Quillen left adjoint it commutes with colimits and preserves cofibrations so it takes the Reedy cofibrant $\widetilde{X}_\bullet$ to the Reedy cofibrant diagram $L\widetilde{X}_\bullet$. Since being cofibrant is invariant under isomorphism the result follows. □

For a T-algebra $X \in \mathscr{C}_T$, we have a Δ_0^{op}-shaped diagram

$$T^\bullet UX\colon \Delta_0^{\mathrm{op}} \to \mathscr{C}$$

where

$$(T^\bullet UX)_n = T^n UX$$

and the maps $(T^\bullet UX)(s_i)$ and $(T^\bullet UX)(d_i)$ are defined as in the bar construction.

Definition 3.18. Suppose that T is a Quillen monad acting on $\mathscr{C}$ and X is a T-algebra. A *bar cofibrant* replacement of X is a cofibrant replacement $\widetilde{X} \to X$ in T-algebras such that $T^\bullet U\widetilde{X}$ is Reedy cofibrant in $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$. We will say X is *resolvable* if it admits a bar cofibrant replacement.

Proposition 3.19. Let T be a Quillen monad acting on $\mathscr{C}$ and $\widetilde{X} \to X$ a bar cofibrant replacement of a T-algebra X. Then the bar resolution $B_\bullet\widetilde{X}$ is a Reedy cofibrant simplicial T-algebra and any two choices of bar cofibrant replacement yield weakly equivalent bar resolutions.

Proof. By assumption $T^\bullet U\widetilde{X}$ is Reedy cofibrant in $\mathscr{C}^{\Delta_0^{\mathrm{op}}}$. By applying the left Quillen functor $F_T\colon \mathscr{C} \to \mathscr{C}_T$ levelwise to this diagram and using Proposition 3.17 we see that $B_\bullet\widetilde{X}$ is Reedy cofibrant.

Two different bar cofibrant resolutions $\widetilde{X}_1$ and $\widetilde{X}_2$ of X are, in particular, cofibrant replacements for X in T-algebras. It follows that there is a weak equivalence of T-algebras $f\colon \widetilde{X}_1 \to \widetilde{X}_2$ which induces a map of bar resolutions. To see that the map is a levelwise weak equivalence we argue by induction. The weak equivalence in degree 0 is induced by applying F_T to the weak equivalence $U\widetilde{X}_1 \xrightarrow{Uf} U\widetilde{X}_2$. Since $\widetilde{X}_1$ and $\widetilde{X}_2$ are bar cofibrant this is a weak equivalence between two cofibrant objects of $\mathscr{C}$. It follows that the induced map in degree 0 is a weak equivalence between two cofibrant T-algebras. The induction argument is similar. □

The remainder of this section is devoted to proving various technical results which will assist in determining when a T-algebra is resolvable.

3.3.1. Monads on diagrams of simplicial sets.

Lemma 3.20. Let S be a set and $\mathscr{C} = s\mathcal{S}et^S$ equipped with the product model structure. Then any diagram $X_\bullet\colon \Delta_0^{\mathrm{op}} \to \mathscr{C}$ is Reedy cofibrant.

Proof. We observe that Δ_0 is *Eilenberg-Zilber* [11, Def. 4.1], i.e., Δ_0 satisfies:

(EZ1) For all surjections $\sigma\colon [n+m]\to[n]$ in Δ_0, the set of sections

$$\Gamma(\sigma)=\{\tau\in\Delta_0 \mid \sigma\tau=\mathrm{id}_{[n]}\}$$

is nonempty.

(EZ2) For any two distinct surjections $\sigma_1,\sigma_2\colon [n+m]\to[n]$, the sets of sections $\Gamma(\sigma_1)$ and $\Gamma(\sigma_2)$ are distinct.

The two conditions are verified as follows: For any surjection σ in Δ_0 consider the section $\sigma'(j)=\min(\sigma^{-1}(j))$. It is immediate that σ' is also in Δ_0 and that $\sigma'_1\neq\sigma'_2$ if $\sigma_1\neq\sigma_2$.

By [11, Prop. 4.2], every Eilenberg-Zilber Reedy category is *elegant* ([11, Def. 3.5]) and by [11, Prop. 3.15] the product and Reedy model structures agree on categories of elegant diagrams in $\mathscr{C}=s\mathcal{S}et^S$. In particular, the object $X_\bullet$ will be Reedy cofibrant because the cofibrations in the product model structure are the levelwise cofibrations and every simplicial set is cofibrant. □

Proposition 3.21. If T is a simplicial Quillen monad acting on $s\mathcal{S}et^S$, then any T-algebra admits a bar cofibrant replacement.

Proof. By Lemma 3.20 any cofibrant replacement in $s\mathcal{S}et^S_T$ is bar cofibrant. □

3.3.2. Cellular monads.

Proposition 3.22. Let $\mathscr{C}$ be a cofibrantly generated model category in which relative cell complexes are monomorphisms and let $X_\bullet\in\mathscr{C}^{\Delta_0^{\mathrm{op}}}$ be a degreewise cellular diagram such that each degeneracy s_i is a subcellular inclusion. Then the latching maps of $X_\bullet$ are cellular inclusions and therefore $X_\bullet$ is Reedy cofibrant.

Proof. The proof of [18, Thm. X.2.7] can be modified to show inductively that [18, (X.2.5)] is a pushout-pullback diagram of subcell complexes defined as unions of the subcell complexes given by the degeneracies. Such unions are well-defined because relative cell complexes are monomorphisms [23, Prop. 10.6.10]. □

Proposition 3.23. Let T be a Quillen monad acting on a cofibrantly generated model category $\mathscr{C}$. Suppose that relative cell complexes in $\mathscr{C}$ are monomorphisms and that for any cellular object M, TM is cellular and the natural unit map $M\to TM$ is a cellular inclusion. If $\widetilde{X}\to X$ is a cofibrant replacement of a T-algebra X, such that $U\widetilde{X}$ is cellular, then $\widetilde{X}$ is a bar cofibrant replacement of X.

Proof. This is an immediate application of Proposition 3.22. □

3.3.3. Monads whose unit maps are inclusions of summands.

Proposition 3.24. Let $\mathscr{C}$ be a pointed model category and let $X_\bullet\in\mathscr{C}^{\Delta_0^{\mathrm{op}}}$ be a diagram such that X_0 is cofibrant and each degeneracy s_i is a cofibration and the inclusion of a summand. Then the latching maps of $X_\bullet$ are cofibrations and summand inclusions, and therefore $X_\bullet$ is Reedy cofibrant.

Proof. First let $X_\infty=\mathrm{colim}_i X_i$ where X_i maps to X_{i+1} via s_0. Let A_∞ be a set such that

$$X_\infty=\bigvee_{\alpha\in A_\infty} Y_\alpha$$

where Y_α cannot be written as a nontrivial coproduct. Since each X_n is an inclusion of a summand of X_{n+1}, there are subsets $A_n \subset A_\infty$ such that

$$X_n = \bigvee_{\alpha \in A_n} Y_\alpha.$$

The degeneracies $X_{n-i} \to X_n$ are summand inclusions and therefore are induced by subset inclusions $A_{n-i} \to A_n$ which we call the set-level degeneracies. We can now identify the degenerate simplices $X_n^{\mathrm{dg}} = \bigvee_{\alpha \in A'_n} Y_\alpha$ where A'_n is the union (i.e., colimit) of all the A_{n-i} under these set-level degeneracies for $1 \le i \le n$. This union of sets indexes the colimit of objects yielding the latching object, so we can identify X_n^{dg} with $L_n X_\bullet$ and the latching map with that induced by the inclusion $A'_n \to A_n$.

Let X_n^{nd} be the complementary summand of $L_n X_\bullet$—this is the nondegenerate part of X_n. To see that the latching map is a cofibration we begin by observing that it is a coproduct of the identity map on the latching object with the map from the initial object into X_n^{nd}. Now X_n is cofibrant because each of the degeneracies are cofibrations and X_0 is cofibrant. The retract X_n^{nd} is therefore cofibrant, and hence the latching map is a coproduct of cofibrations. □

Proposition 3.25. Let T be a simplicial Quillen monad acting on a pointed simplicial model category $\mathscr{C}$. Suppose that for any cofibrant object M the natural unit map $M \to TM$ is a cofibration and inclusion of a summand. If $\widetilde{X} \to X$ is a cofibrant replacement of a T-algebra X, such that $U\widetilde{X}$ is cofibrant, then $\widetilde{X}$ is a bar cofibrant replacement of X.

Proof. This is an immediate application of Proposition 3.24. □

4. The spectral sequence and examples

4.1. Proof of Theorem A

Now we recall and prove the central theorem of this paper:

Theorem A. Let $\mathscr{C}$ be a simplicial model category and T a simplicial Quillen monad acting on $\mathscr{C}$. Let X and Y be T-algebras. Suppose that

a. T commutes with geometric realization and
b. X is resolvable with bar cofibrant replacement $\widetilde{X} \to X$.

Let $U\colon \mathscr{C}_T \to \mathscr{C}$ denote the forgetful functor from the category of T-algebras to $\mathscr{C}$. Then T induces a monad hT on $ho\mathscr{C}$ and there exists an obstruction-theoretic spectral sequence satisfying:

1. $E_1^{0,0} = ho\mathscr{C}(UX, UY)$.
2. $E_2^{0,0} = (ho\mathscr{C})_{hT}(UX, UY)$. That is, a homotopy class $[f]\colon UX \to UY$ survives to the E_2 page if and only if it is a map of hT-algebras in the homotopy category.
3. Provided a T-algebra map $\varepsilon\colon X \to Y$ to serve as a base point, the spectral sequence conditionally converges to the homotopy of the derived mapping space

$$\pi^s \pi_t(\mathscr{C}^d(T^\bullet U\widetilde{X}, UY), \varepsilon) \Longrightarrow \pi_{t-s}(\mathscr{C}_T^d(X,Y), \varepsilon).$$

4. In this case the differentials $d_r[f]$ provide obstructions to lifting $[f]$ to a map of T-algebras.

5. The edge homomorphisms

$$\begin{aligned}\pi_0\mathscr{C}_T^d(X,Y) \twoheadrightarrow E_\infty^{0,0} & \\ & \hookrightarrow E_2^{0,0} = (ho\mathscr{C})_{hT}(UX,UY) \\ & \hookrightarrow E_1^{0,0} = ho\mathscr{C}(UX,UY)\end{aligned}$$

   are the corresponding forgetful functors.

6. If $\mathscr{C}_T$ has functorial bar cofibrant and fibrant replacements, then the spectral sequence is contravariantly functorial in X and covariantly functorial in Y.

7. A map of simplicial monads $T_1 \to T_2$ satisfying the above hypotheses induces a contravariant map of spectral sequences provided that X has a bar cofibrant replacement $\widetilde{X} \to X$ in $\mathscr{C}_{T_2}$ such that $U_3\widetilde{X}$ has a bar cofibrant replacement in $\mathscr{C}_{T_1}$.

Proof. First, in order for the theorem to make sense there must be a derived mapping space of T-algebras and this follows from the assumption that T is simplicial Quillen.

The conclusions of the theorem depend only on the weak equivalence classes of X and Y, so without loss of generality we assume Y is a fibrant T-algebra and, replacing X with $\widetilde{X}$ if necessary, that X is a bar cofibrant T-algebra. By Proposition 3.19 the bar resolution $B_\bullet X$ is a Reedy cofibrant simplicial T-algebra and any two choices of bar cofibrant replacement yield equivalent bar resolutions. Since Y is fibrant and $\mathscr{C}$ is a simplicial model category, applying the mapping space functor $\mathscr{C}_T(-,Y)$ to a Reedy cofibrant simplicial T-algebra yields a Reedy fibrant cosimplicial space. In particular, $\mathscr{C}_T(B_\bullet X, Y)$ is Reedy fibrant.

Applying [16], the totalization tower for this Reedy fibrant cosimplicial space arising from the skeletal filtration on $|B_\bullet X|$ yields an obstruction-theoretic spectral sequence computing the homotopy of the totalization

$$\mathrm{Tot}(\mathscr{C}_T(B_\bullet X, Y)) \cong \mathscr{C}_T(|B_\bullet X|, Y).$$

This spectral sequence conditionally converges provided there exists a base point at which to take homotopy groups. (A list of obstructions to determining such a base point is also provided by the construction; see Remark 4.4.)

Now since $B_\bullet X$ is Reedy cofibrant and $\mathscr{C}_T$ is a simplicial model category, $|B_\bullet X|$ is a cofibrant T-algebra. Since T commutes with geometric realization, Proposition 3.13 shows that the augmentation map

$$|B_\bullet X| \to X$$

is a weak equivalence of T-algebras. It follows that $\mathscr{C}_T(|B_\bullet X|, Y)$ is a model for $\mathscr{C}_T^d(X,Y)$ and this gives the target of the spectral sequence in (3). Conclusion (4) follows immediately from the conditional convergence of the spectral sequence.

The $E_1^{0,0}$ term of the Bousfield-Kan spectral sequence is the set

$$\pi_0\mathscr{C}_T(B_0X, Y) = \pi_0\mathscr{C}_T(F_TUX, Y) \cong \pi_0\mathscr{C}(UX, UY).$$

To prove (1) we will show the right-hand side can be identified with morphisms in the homotopy category. This follows if UX is cofibrant and UY is fibrant since $\mathscr{C}$ is a simplicial model category. These conditions follow from the hypotheses that X is bar cofibrant and that T is Quillen: $T^\bullet UX$ is Reedy cofibrant so the zeroth latching map shows that UX is cofibrant. Since T is Quillen, U is a right Quillen functor and therefore UY is fibrant because Y is fibrant.

The edge homomorphism

$$\pi_0\mathscr{C}^d_T(X,Y) \to E_1^{0,0}$$

is induced by restricting along the inclusion

$$\mathrm{sk}_0|B_\bullet X| = F_T U X \to |B_\bullet X|$$

which by adjunction gives the second half of (5). The first half will follow from the identification of the $E_2^{0,0}$ term in (2).

To prove (2) recall that the $E_2^{0,0}$ term of the Bousfield-Kan spectral sequence is defined to be the equalizer of the two face maps

$$\pi_0\mathscr{C}_T(B_0X,Y) \rightrightarrows \pi_0\mathscr{C}_T(B_1X,Y).$$

We again use the adjunction and the fact that $T^\bullet UX$ is Reedy cofibrant to see that the diagram above is isomorphic to

$$ho\mathscr{C}(UX,UY) \rightrightarrows ho\mathscr{C}(TUX,UY),$$

whose equalizer is, by definition, $(ho\mathscr{C})_{hT}(UX,UY)$ (see (2.2) and Proposition 3.7). In other words, a map lifts to $E_2^{0,0}$ precisely if it is a homotopy T-algebra map.

Provided a base point ϵ for the spectral sequence, or even a point that lifts to Tot^2 (see Remark 4.4), the E_1 page of this spectral sequence is given by applying π_t to the spaces $\mathscr{C}_T(B_sX,Y)$ and normalizing as in [16, § 2.4]. The E_2 term can be identified with the cohomotopy of this graded cosimplicial object which is typically denoted as follows:

$$E_2^{s,t} = \pi^s\pi_t(\mathscr{C}_T(B_\bullet X,Y),\epsilon).$$

By adjunction we have

$$\mathscr{C}_T(B_nX,Y) = \mathscr{C}_T(F_TT^nUX,Y) \cong \mathscr{C}(T^nUX,UY).$$

As in the previous steps, the right-hand side is a model for the derived mapping space since UY is fibrant and $T^\bullet UX$ is Reedy cofibrant. This completes the proof of (3).

Since the Bousfield-Kan spectral sequence is functorial in maps of fibrant cosimplicial spaces, to prove (6) it suffices to see that our construction of the fibrant cosimplicial space is functorial in X and Y. This is immediate from the functoriality of the bar resolution and the conditions of (6).

To prove (7) we note that a map of simplicial monads $T_1 \to T_2$ between cocomplete categories determines a monadic adjunction (F_{T_3},U_3) fitting into the following diagram [15, Cor. 4.57, Prop. 4.5.9]:

$$\begin{tikzcd}
\mathscr{C}_{T_1} \arrow[rr, shift left, "F_{T_3}"] \arrow[dr, shift left, "U_1"] & & \mathscr{C}_{T_2} \arrow[ll, shift left, "U_3"] \arrow[dl, shift left, "U_2"] \\
& \mathscr{C} \arrow[ul, shift left, "F_{T_1}"] \arrow[ur, shift left, "F_{T_2}"] &
\end{tikzcd}$$

To see that U_3 induces a morphism of simplicial mapping objects, observe that a map of simplicial monads induces a natural transformation between the equalizer diagrams in simplicial sets which determine the simplicial mapping objects in T_2 and T_1-algebras respectively (cf. (2.2)). Since both categories of algebras inherit their composition laws from the composition in $\mathscr{C}$ we see that U_3 is a simplicial functor.

To obtain a map between the spectral sequences corresponding to

$$\mathscr{C}_{T_2}(X,Y) \xrightarrow{U_3} \mathscr{C}_{T_1}(U_3X,U_3Y)$$

let $r\colon (U_3X)^b \to U_3X$ be a bar cofibrant replacement of U_3X in $\mathscr{C}_{T_1}$. We remind the reader that we have already replaced X and Y by their bar cofibrant and fibrant replacements respectively. Now apply U_3 to the T_2-bar resolution for X, precompose with the unit map $e\colon \mathrm{Id}_{\mathscr{C}_{T_1}} \to T_3$, and take bar cofibrant and fibrant replacements of U_3X and U_3Y respectively[2]:

$$\begin{aligned}
\mathscr{C}_{T_2}(B_\bullet(F_{T_2},T_2,U_2X),Y) \xrightarrow{U_3} & \ \mathscr{C}_{T_1}(U_3B_\bullet(F_{T_2},T_2,U_2X),U_3Y)\\
&= \mathscr{C}_{T_1}(B_\bullet(T_3F_{T_1},U_1T_3F_{T_1},U_1U_3X),U_3Y)\\
&\xrightarrow{e^*} \mathscr{C}_{T_1}(B_\bullet(F_{T_1},T_1,U_1U_3X),U_3Y)\\
&\xrightarrow{r^*} \mathscr{C}_{T_1}(B_\bullet(F_{T_1},T_1,U_1(U_3X)^b),(U_3Y)^f).
\end{aligned}$$

So by our assumptions on X, we see that U_3 induces a morphism between two fibrant cosimplicial spaces whose associated spectral sequences are the T-algebra spectral sequences calculating $\pi_*\mathscr{C}^d_{T_2}(X,Y)$ and $\pi_*\mathscr{C}^d_{T_1}(U_3X,U_3Y)$ respectively. □

We highlight two immediate corollaries of Theorem A.

Corollary 4.1. The forgetful functor taking a non-empty $ho(\mathscr{C}_T)(X,Y)$ to $(ho\mathscr{C})_{hT}(X,Y)$ is surjective if and only if the differential d_r on $E_r^{0,0}$ is trivial for all $r \geq 2$.

Corollary 4.2. Suppose the portion of the spectral sequence computing $\pi_0\mathscr{C}^d_T(X,Y)$ converges [16, § 4.2], i.e., there exists a base point ε and

$$\lim_s{}^1 \pi_1(\mathscr{C}^d_T(\mathrm{sk}_s|B_\bullet\widetilde{X}|,Y),\varepsilon) = 0.$$

Then the forgetful functor taking $ho(\mathscr{C}_T)(X,Y)$ to $(ho\mathscr{C})_{hT}(X,Y)$ is injective if and only if $E_\infty^{t,t} = 0$ for $t > 0$.

Remark 4.3. As stated in [16], every entry in the spectral sequence above should consist of pointed sets. We have chosen to omit the distinguished point $[\varepsilon]$ in bidegree $(0,0)$ to simplify the statement of Theorem A.

Remark 4.4. There are, in fact, a variety of obstruction sequences whose vanishing can give a lift of ε through the totalization tower. The following are special cases of [16, §§ 2.4, 2.5, 5.2] for a cosimplicial object $X_\bullet$ in a simplicial category $\mathscr{D}$:

1. The rth spectral sequence page $E_r^{p,q}$ is defined if there is an element $\varepsilon_{r-1} \in \mathrm{Tot}^{r-1}\mathscr{D}(X_\bullet,Y)$ which lifts to $\mathrm{Tot}^{2r-2}\mathscr{D}(X_\bullet,Y)$, and the page depends naturally on ε_{r-1}.

[2] In practice, all of the model structures on T-algebras are usually right induced and U_3 is necessarily a simplicial right Quillen functor. In this case taking a fibrant replacement is not necessary.

2. Let $\varepsilon_p \in \mathrm{Tot}^p\,\mathscr{D}(X_\bullet, Y)$, and let ε_k be the projection of ε_p to $\mathrm{Tot}^k\,\mathscr{D}(X_\bullet, Y)$. If

$$p/2 \le k \le p$$

then there is an obstruction element lying in $E^{p+1,p}_{p-k+1}$ which vanishes if and only if ε_k lifts to $\mathrm{Tot}^{p+1}\,\mathscr{D}(X_\bullet, Y)$.

If Whitehead products vanish in each $\mathscr{D}(X_s, Y)$ (e.g., when the mapping spaces of $\mathscr{D}$ are H-spaces), then the range in which the obstruction classes are defined can be extended as follows:

1′. The rth spectral sequence page $E^{p,q}_r$ is defined if there is an element $\varepsilon_{r-2} \in \mathrm{Tot}^{r-2}\,\mathscr{D}(X_\bullet, Y)$ which lifts to $\mathrm{Tot}^{2r-3}\,\mathscr{D}(X_\bullet, Y)$, and the page depends naturally on ε_{r-2}.

2′. Let $\varepsilon_p \in \mathrm{Tot}^p\,\mathscr{D}(X_\bullet, Y)$, and let ε_k be the projection of ε_p to $\mathrm{Tot}^k\,\mathscr{D}(X_\bullet, Y)$. If

$$(p-1)/2 \le k \le p$$

then there is an obstruction element lying in $E^{p+1,p}_{p-k+1}$ which vanishes if and only if ε_k lifts to $\mathrm{Tot}^{p+1}\,\mathscr{D}(X_\bullet, Y)$.

Taking $p = 1$ and $k = 0$ in (2′) from Remark 4.4, we obtain the following useful refinement of Theorem A.

Theorem 4.5. (Cf. [20, Cor. 2.4.15]) Let $\mathscr{C}$ be a simplicial model category and T a simplicial Quillen monad acting on $\mathscr{C}$ such that T commutes with geometric realization. Let X and Y be T-algebras, such that X admits a bar cofibrant replacement $\widetilde{X}$. Moreover, assume that the derived mapping spaces $\mathscr{C}^d(T^n U\widetilde{X}, UY)$ have the homotopy type of H-spaces.

Then the T-algebra spectral sequence exists, its E_2 term is always defined, and there is a series of successively defined obstructions to realizing a map

$$[f] \in E_2^{0,0} = (ho\mathscr{C})_{hT}(UX, UY)$$

in the groups

$$E_2^{s+1,s} \cong \pi^{s+1}\pi_s(\mathscr{C}^d(T^\bullet U\widetilde{X}, UY), f)$$

for $s \ge 1$. In particular, if these groups are always zero, then the map induced by the forgetful functor

$$ho(\mathscr{C}_T)(X,Y) \to (ho\mathscr{C})_{hT}(UX, UY)$$

is surjective. If the portion of the spectral sequence computing $\pi_0\mathscr{C}^d_T(X,Y)$ converges and

$$\pi^s\pi_s(\mathscr{C}^d(T^\bullet U\widetilde{X}, UY), f) = 0$$

for each choice of $[f]$ and all $s \ge 1$, then this map is an injection.

4.2. Observations on E_1

To simplify notation for this section we assume that $\mathscr{C}$, T, X, and Y are as in Theorem A and we have replaced X with $\widetilde{X}$ and Y with its fibrant replacement if necessary.

Provided all of the terms in $E_1^{s,t}$ of the T-algebra spectral sequence for $t > 0$ are abelian groups, then we can avoid using the normalized cocomplex in [16] and instead use Moore cochains. For example, this happens if the mapping spaces $\mathscr{C}(T^n UX, UY)$ have the homotopy type of H-spaces

s	-1	0	1
2	$ho\mathscr{C}_{\downarrow UY}\left(T^2UX, UY^{S^1}\right)$	$ho\mathscr{C}_{\downarrow UY}\left(T^2UX, UY^{S^2}\right)$	$ho\mathscr{C}_{\downarrow UY}\left(T^2UX, UY^{S^3}\right)$
1	$ho\mathscr{C}(TUX, UY)$	$ho\mathscr{C}_{\downarrow UY}\left(TUX, UY^{S^1}\right)$	$ho\mathscr{C}_{\downarrow UY}\left(TUX, UY^{S^2}\right)$
0		$ho\mathscr{C}(UX, UY)$	$ho\mathscr{C}_{\downarrow UY}\left(UX, UY^{S^1}\right)$
			$t-s$

Figure 4.6: Low-degree terms on the E_1 page of the T-algebra spectral sequence, interpreted as homotopy classes of lifts.

(Theorem 4.5). For the unnormalized complex, we can apply the tensor-cotensor adjunction to obtain the following identification of $E_1^{s,t}$:

$$E_1^{s,t} = \pi_t\big(\mathscr{C}(T^sUX, UY), \varepsilon\big) \cong \pi_0 \mathscr{C}_{\downarrow UY}(T^sUX, UY^{S^t}).$$

This displays $E_1^{s,t}$ as a set of homotopy classes of lifts in the diagram below, with homotopies over ε:

$$\begin{tikzcd} & UY^{S^t} \arrow[d] \\ T^sUX \arrow[r, "\varepsilon"] \arrow[ur, dashed] & UY \end{tikzcd}$$

As in the proof of Theorem A, $T^\bullet UX$ is Reedy cofibrant and it follows that each T^nUX is cofibrant in $\mathscr{C}$. Since U creates cotensors, preserves fibrations, and Y is fibrant in $\mathscr{C}_T$, we see UY^{S^t} is fibrant in $\mathscr{C}$. Now the overcategory is a cofibrantly generated simplicial model category whose cofibrations/fibrations/weak equivalences are those of $\mathscr{C}$. So these objects are cofibrant and fibrant respectively in $\mathscr{C}_{\downarrow UY}$. Regarding UX as an object over UY by a chosen map $\varepsilon\colon UX \to UY$, we identify the unnormalized complex:

$$E_1^{s,t} \cong ho\mathscr{C}_{\downarrow UY}(T^sUX, UY^{S^t}) \quad \text{for } t > 0.$$

4.3. Theorem B: Quillen cohomology and the E_2-term

The purpose of this section is to prove Theorem B, which gives criteria for obtaining an algebraic description of the E_2 term from Theorem 4.5.

Theorem B. Let T be a simplicial Quillen monad acting on a simplicial model category $\mathscr{C}$ such that T commutes with geometric realization. Let $X, Y \in \mathscr{C}_T$. Suppose that bar cofibrant replacements exist in $\mathscr{C}_T$ so that, without loss of generality, we can assume X and Y are bar cofibrant and fibrant respectively. Moreover, assume that the derived mapping spaces $\mathscr{C}^d(T^n UX, UY)$ have the homotopy type of H-spaces and there is a functor

$$\pi_* : ho\mathscr{C} \to \mathscr{D}$$

such that

a. The associated map

$$\pi_* : ho\mathscr{C}(T^s UX, UY^{S^t}) \to \mathscr{D}(\pi_* T^s UX, \pi_* UY^{S^t})$$

is an isomorphism for all $s, t \geq 0$.

b. There is a natural isomorphism $\pi_* TX \cong T_{\text{alg}} \pi_* X$ for a monad T_{alg} compatible with the structure homomorphisms of T and T_{alg}.

c. The categories $\mathscr{D}$ and $\mathscr{D}_{T_{\text{alg}}}$ are categories of $\mathcal{S}et$-valued models for some graded algebraic theories (i.e., they are algebraic categories in the sense of Quillen).

d. For $t \geq 1$, $\pi_* Y^{S^t}$ is naturally an abelian group object in the category of T_{alg}-algebras over $\pi_* Y$.

Then the E_2 term of the T-algebra spectral sequence exists and can be identified as follows:

$$\begin{aligned} E_2^{0,0} &\cong \mathscr{D}_{T_{\text{alg}}}(\pi_* UX, \pi_* UY) \\ E_2^{s,t} &\cong H^s_{Q,\pi_* UY}(\pi_* UX; \pi_* UY^{S^t}) \quad \text{for } t > 0. \end{aligned}$$

Here the cohomology groups on the second line are the associated Quillen cohomology groups of our T_{alg}-algebra $\pi_* X$ viewed as an algebra over $\pi_* Y$ via a choice of an element in $E_2^{0,0}$ as in Definition 3.8.

Proof. First we identify $E_2^{0,0}$: As remarked in the proof of Theorem A, this is computed by the equalizer

$$E_2^{0,0} = \operatorname{eq}\big(ho\mathscr{C}(UX, UY) \rightrightarrows ho\mathscr{C}(TUX, UY)\big)$$

where the morphisms are induced by the T-algebra structures on X and Y respectively. Using the isomorphisms from condition (a) we obtain the identification

$$E_2^{0,0} \cong \operatorname{eq}\big(\mathscr{D}(\pi_* UX, \pi_* UY) \rightrightarrows \mathscr{D}(\pi_* TUX, \pi_* UY)\big).$$

Applying π_* and condition (b) we obtain

$$E_2^{0,0} \cong \mathscr{D}_{T_{\text{alg}}}(\pi_* UX, \pi_* UY) \cong \operatorname{eq}\big(\mathscr{D}(\pi_* UX, \pi_* UY) \rightrightarrows \mathscr{D}(T_{\text{alg}} \pi_* UX, \pi_* UY)\big).$$

Now to identify the remainder of the E_2 term we pick a map $f : UX \to UY$ representing some element $[f] \in E_2^{0,0}$. By Theorem A for $t > 0$

$$E_2^{s,t} \cong \pi^s \pi_t\big(\mathscr{C}(T^\bullet UX, UY), f\big).$$

Since the hypotheses of Theorem 4.5 are satisfied

$$\pi_t\big(\mathscr{C}(T^\bullet UX, UY), f\big)$$

is a cosimplicial abelian group for $t > 0$ and the cohomotopy group π^s can be calculated as the sth cohomology group of the associated Moore cochain complex. By the discussion in Section 4.2 we see

$$\pi_t\big(\mathscr{C}(T^s UX, UY), f\big) \cong ho\mathscr{C}_{\downarrow UY}(T^s UX, UY^{S^t}).$$

Applying the homotopy invariant functor π_* and conditions (a) and (b) we obtain

$$\begin{aligned} E_2^{s,t} &\cong H^s\Big(\mathscr{D}_{\downarrow \pi_* Y}(T^\bullet_{\mathrm{alg}}\pi_* UX, \pi_* UY^{S^t})\Big) \\ &\cong H^s\Big(\mathscr{D}_{T_{\mathrm{alg}}\downarrow \pi_* Y}(F_{T_{\mathrm{alg}}} T^\bullet_{\mathrm{alg}}\pi_* UX, \pi_* UY^{S^t})\Big) \end{aligned}$$

where the last isomorphism uses the fact that $\pi_* UY^{S^t}$ is a T_{alg}-algebra over $\pi_* Y$. These cohomology groups are, by definition, the *cotriple cohomology groups* of $\pi_* UX$ with respect to the cotriple associated to the monad T_{alg}.

To complete the proof we identify the cotriple cohomology with Quillen cohomology using (c), (d), and [40, § II.5 Thm. 5]. □

4.4. Applicable contexts

This section will be devoted to demonstrating that the hypotheses of Theorem A are satisfied in many categories of interest.

4.4.1. Simplicial algebraic theories

Theorem 4.7. If T is a monad on $s\mathcal{S}et^S$ associated to an S-graded algebraic theory as in Section 2.2, then the T-algebra spectral sequence can be applied functorially to any $X, Y \in s\mathcal{S}et^S_T$.

Proof. By Proposition 3.4 we see that T is simplicial Quillen and $s\mathcal{S}et^S_T$ is a cofibrantly generated model category. Applying Proposition 3.11 with the decomposition from (2.9) shows that T commutes with geometric realizations. Finally since $s\mathcal{S}et^S_T$ is cofibrantly generated, it admits functorial cofibrant and fibrant replacements. It follows from Proposition 3.21 that $s\mathcal{S}et^S_T$ admits functorial bar cofibrant replacements. □

For example, by Remark 2.19 we can apply the T-algebra spectral sequence to analyze spaces of operad maps. Since the space of operad maps from an operad $\mathscr{O}$ to the endomorphism operad of an object X (when defined) is in correspondence with the space of algebra structures on X [43], one could, in principle, use this spectral sequence to analyze algebra structures on X.

4.4.2. G-actions

For the following result one can use any of the standard cofibrantly generated models for the category of spectra which is enriched in spaces, whose relative cell complexes are monomorphisms, and such that the tensor product of a subcellular inclusion of spaces with a cellular spectrum is naturally a subcellular inclusion of spectra.

Proposition 4.8. Let G be a topological group admitting a cellular structure such that the inclusion of the unit $e \to G$ is the inclusion of a sub-complex. Let $TX = G_+ \wedge X = \mathrm{Res}^G_e \mathrm{Ind}^G_e X$ be the monad on pointed spaces/spectra whose algebras are pointed G-spaces/G-spectra. Then the T-algebra spectral sequence can be applied functorially to any X, Y in these categories.

Proof. It is well known and straightforward to show using Theorem 3.2 and Remark 3.3 that T is simplicial Quillen. Since geometric realization commutes with smash products in either of these categories we see that T commutes with geometric realization. Since the unit transformation applied to cellular spectra gives an inclusion of subcomplexes, by Proposition 3.23 we see that the equivariant cellular replacement of X is bar cofibrant. □

In the case of pointed G-spaces or G-spectra the T-algebra spectral sequence takes a familiar form. The bar resolution of a CW-complex X is the standard cofibrant replacement $B(G_+, G_+, X) \to X$ in the 'Borel' or projective model structure on G-objects. We emphasize that, although this is a common model structure to work with, it is not the standard model structure used in equivariant homotopy theory (cf. [37]). In particular, the cofibrant replacement of S^0 in this model structure is EG_+, which is often infinite dimensional.

Note that G acts levelwise on the left of the simplicial bar construction and that the left copy of G_+ in each degree of the bar construction is actually notation for the left adjoint Ind_e^G. The standard equivariant equivalence $\mathrm{Ind}_e^G(Y) \wedge X \cong \mathrm{Ind}_e^G(Y \wedge \mathrm{Res}_e^G X)$ [37, (I.1.6)] induces a G-equivariant isomorphism between $EG_{\bullet +} \wedge X$ with its diagonal G-action and $B_\bullet(G_+, G_+, X)$ with G acting on the left. So the T-algebra spectral sequence computing the homotopy groups of the derived space of G-maps, in the projective model structure, between X and Y becomes a homotopy fixed point spectral sequence. More specifically the spectral sequence computes the homotopy groups of $F(X^c, Y^f)^{hG} \cong F(EG_+ \wedge X^c, Y^f)^G$ where $F(X^c, Y^f)$ is the corresponding G-space of maps and X^c and Y^f are functorial cofibrant and fibrant replacements of X and Y respectively.

Remark 4.9. As one might expect, the homotopy G-spaces/spectra (i.e., the homotopy T-algebras for T as above) will correspond to those spaces/spectra which admit a G-action in the homotopy category. Morphisms of homotopy G-spaces/spectra are maps in the homotopy category which commute with the G-action. In particular, any G-map which is non-equivariantly null-homotopic is necessarily trivial in the category of homotopy G-spaces (see Section 5.1).

4.4.3. Algebras over operads

As in Section 2.4, we will continue to assume that all of our operads are defined in simplicial sets. We will say an operad is cofibrant if it is cofibrant in the model structure associated to the graded simplicial algebraic theory discussed in Remark 2.19. This is the same model structure considered in [9, § 3.3.1].

Suppose that $\mathscr{C}$ is a simplicial symmetric monoidal model category satisfying the hypotheses of Proposition 2.20. As shown in Section 2.4, this assumption implies that the category of $\mathscr{O}$-algebras in $\mathscr{C}$ is equivalent to a simplicial category of $T_{\mathscr{O}}$-algebras. We will say that $\mathscr{O}$ is *admissible* if $T_{\mathscr{O}}$ is a simplicial Quillen monad such that $\mathscr{C}_{T_{\mathscr{O}}}$ has a right induced model structure.

Proposition 4.10. Let $\mathscr{C}$ be a symmetric monoidal simplicial model category satisfying the hypotheses of Proposition 2.20. Let T be the monad associated to a cofibrant admissible operad and suppose that

a. geometric realization commutes with the symmetric monoidal structure on $\mathscr{C}$ and
b. one of the following conditions holds:
   (*i*) The underlying category $\mathscr{C}$ is $s\mathcal{S}et^S$ for some set S.
   (*ii*) $\mathscr{C}$ is cofibrantly generated, relative cell complexes in $\mathscr{C}$ are monomorphisms, for each cellular $X \in \mathscr{C}$ the unit map $X \to TX$ is a cellular inclusion, and the object underlying each cellular T-algebra is a cellular object of $\mathscr{C}$.
   (*iii*) $\mathscr{C}$ is pointed, for each cofibrant $X \in \mathscr{C}$ the unit map $X \to TX$ is a cofibration and the inclusion of a summand, and the the object underlying each cofibrant T-algebra is cofibrant in $\mathscr{C}$.

Then the T-algebra spectral sequence can be applied to any $X, Y \in \mathscr{C}_T$. Moreover if $\mathscr{C}_T$ is cofibrantly generated, then the spectral sequence is functorial in X and Y.

Proof. Proposition 2.20 shows that $\mathscr{C}_T$ is a bicomplete simplicial category. By definition of admissibility, T is simplicial Quillen. Since geometric realization commutes with the monoidal structure, we can apply Proposition 3.11 to (2.18) and see that T commutes with geometric realization.

Since our operad is admissible cofibrant we can replace any T-algebra by one which is cofibrant in $\mathscr{C}$ by the argument in [44, Rem. 13.3] (cf. [9, Thm. 3.5(b)]). Finally by the remaining hypothesis we can apply either Proposition 3.21, Proposition 3.23, or Proposition 3.25 to see that the cofibrant/cellular replacement of any T-algebra is bar cofibrant. $\square$

Proposition 4.11. Let $\mathscr{C}$ be a pointed symmetric monoidal simplicial model category satisfying the hypotheses of Proposition 2.20. Let T be a monad acting on $\mathscr{C}$ and suppose that

a. geometric realization commutes with the symmetric monoidal structure on $\mathscr{C}$ and
b. T arises from an admissible operad $W\mathscr{O}$, where $W\mathscr{O}$ is the Boardman-Vogt cofibrant replacement of an operad $\mathscr{O}$ (see [10]) such that $\mathscr{O}(0) = \mathscr{O}(1) = *$.

Then the T-algebra spectral sequence can be applied to any $X, Y \in \mathscr{C}_T$.

Proof. We will apply Proposition 4.10 using the hypotheses that $\mathscr{C}$ is pointed and that the unit map is a cofibration and the inclusion of a summand. As shown in [10] the Boardman-Vogt construction yields a functorial cofibrant replacement of our operad. We will show in Lemma 4.12 that $(W\mathscr{O})(1) = *$, so the unit map $X \to TX$ is always the inclusion of a summand.

Since $W\mathscr{O}$ is cofibrant, by replacing X with a cofibrant replacement if necessary, we can assume X is cofibrant in $\mathscr{C}$ by the argument in Proposition 4.10. Since $\mathscr{C}$ is a symmetric monoidal model category it is straightforward to apply the pushout-product axiom and induction on n to see that $X^{\otimes n}$ is cofibrant. Finally, since our cofibrant operad is Σ-cofibrant [10, § 2.4] we see that $W\mathscr{O}(n) \otimes X^{\otimes n}$ is a retract of a cellular complex built with free Σ_n-cells. It follows that $W\mathscr{O}(n) \otimes_{\Sigma_n} X^{\otimes n}$ is cofibrant which in turn implies TX is cofibrant. $\square$

Lemma 4.12. Suppose that $\mathscr{O}$ is an operad in $s\mathcal{S}et$ such that $\mathscr{O}(0) = \mathscr{O}(1) = *$. Then $W\mathscr{O}(1) = *$.

Proof. The result follows immediately from the construction of $W\mathscr{O}$ in [10] and we use the notation therein. Namely, under the given hypotheses all of the maps in the sequential colimit

$$W(H, \mathscr{O})(n) = \operatorname{colim}\left(\mathscr{O}(n) = W_0(H, \mathscr{O})(n) \to W_1(H, \mathscr{O})(n) \to \cdots\right)$$

are isomorphisms when $n = 1$ (H is the unit interval here). To see this, one observes that the right-hand (and therefore left-hand) vertical maps in the pushout [10, (13)] are isomorphisms for $n = 1$: For trees G with a single input edge, the objects $\underline{\mathscr{O}}(G)$ and $\underline{\mathscr{O}}^-(G)$ are equal (all vertices of G are univalent, and if $\mathscr{O}(1) = *$ then $\underline{\mathscr{O}}_c(G) = \underline{\mathscr{O}}(G)$ for any subset of univalent vertices c). As an aside, note that this implies the vertical arrows in the pushout diagram at the end of [10, § 3] are isomorphisms for $n = 1$, and hence $\mathscr{F}_*(\mathscr{O})(1) = \mathscr{O}(1) = *$. Moreover, this implies $(H \otimes \mathscr{O})^-(G) = H(G) \otimes \mathscr{O}^-(G)$. Therefore the vertical maps in [10, (13)] are isomorphisms and $W(H, \mathscr{O})(1) = W_0(H, \mathscr{O})(1) = \mathscr{O}(1)$. $\square$

Let R be a commutative ring spectrum. For the following corollary one can use any symmetric monoidal category of R-modules satisfying the conditions of Proposition 2.20 and condition (a) of Proposition 4.11. These conditions are easily verified in the standard cases such as those of [18, 26, 35].

Corollary 4.13. Suppose T is the monad associated to the Boardman-Vogt replacement of either the associative or the commutative operad (so it is an A_∞ or E_∞ operad) acting on $\mathcal{M}od_R$. Then the T-algebra spectral sequence can be applied to any T-algebras X and Y.

5. Computations

5.1. G-actions

The next two examples provide, respectively, an example of a non-trivial G-map which is trivial as a homotopy G-map and an example of a non-trivial homotopy G-map which does not lift to a G-map. We emphasize that we are working in the projective model structure for G-objects discussed in Section 4.4.2 and as a consequence the derived space of equivariant maps is modeled by the homotopy fixed points of the underlying derived space of maps.

Example 5.1. Regard $\mathbb{R}$ as a C_2-space via the sign action. Then applying one point compactification to the inclusion

$$\{0\} \to \mathbb{R}$$

yields an essential map

$$e_\sigma : S^0 \to S^\sigma$$

of pointed C_2-spaces.

We use the trivial map as a base point for the T-algebra spectral sequence computing the derived space of pointed C_2-equivariant maps between S^0 and S^σ. We can identify the E_2 term as follows: [3]:

$$E_2^{s,t} = H^s(C_2; \pi_t S^\sigma) \Longrightarrow \pi_{t-s}(S^\sigma)^{hC_2}.$$

As noted in Remark 4.9, e_σ must represent the trivial map in the category of pointed homotopy C_2-spaces. The spectral sequence confirms this since $E_2^{0,0} = (\pi_0 S^\sigma)^{C_2} = 0$. In fact, since the homotopy groups of S^1 are concentrated in degree 1, this spectral sequence is concentrated on the line $t = 1$ and necessarily collapses at E_2. Computing the group cohomology with coefficients $\pi_1(S^\sigma) \cong \mathbb{Z}$ twisted by the sign action, we see that the only non-zero contribution is from $E_2^{1,1} = \mathbb{Z}/2$, which detects the essential map e_σ above.

Example 5.2. Let C_2 act on KU via complex conjugation. The C_2-action on $\pi_* KU$ is trivial precisely on those homotopy groups generated by even powers of the Bott map. In particular, if we regard S^4 as having a trivial C_2 action we obtain a non-trivial map

$$\beta^2 : S^4 \to KU$$

in the category of homotopy C_2-spectra.

The T-algebra spectral sequence computing the homotopy groups of the space of C_2-equivariant maps from S^0 to KU is a connective cover of the homotopy fixed point spectral sequence (see Figure 5.3). The latter spectral sequence converges to the homotopy of KO and there is a well-known differential $d_3(\beta^2) = \eta^3$ (see [45, Prop. 5.3.1]) which follows from a comparison with the Adams-Novikov spectral sequence and the relation $\eta^4 = 0$ in $\pi_* S$. Since the T-algebra spectral sequence computing $\pi_* \mathcal{S}pectra^d_{C_2}(S^4, KU)$ is just a shift of the T-algebra spectral sequence computing $\pi_* \mathcal{S}pectra^d_{C_2}(S^0, KU)$, we see that the element $\beta^2 \in E_2^{0,0}$ supports a d_3 and does not lift to a map of C_2-spectra.

[3]Normally instability, e.g., actions of the fundamental group, prevents getting such a simple description of the E_2 term, however in this case S^σ is non-equivariantly an Eilenberg-MacLane space for $\mathbb{Z}$ and so the second half of the refinements in Remark 4.4 apply.

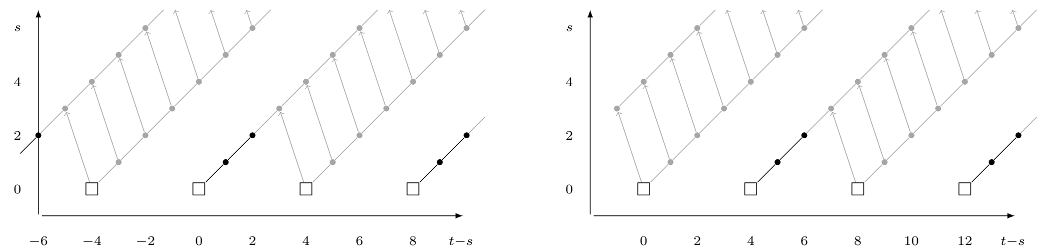

Figure 5.3: Homotopy fixed-point spectral sequence for $\pi_*(KU^{hC_2})$ and the T-algebra spectral sequence for $\pi_*\mathcal{S}pectra^d_{C_2}(S^4, KU)$.

5.2. Algebras over an operad in spectra

A_∞-algebras in Hk-modules

For one example of Theorem B in action, let k be a field and T the monad

$$X \mapsto TX = \bigvee_{n\geq 0} K_n \otimes X^{\wedge_{Hk} n}$$

on Hk-module spectra associated to the A_∞ operad. Conditions (a) and (b) from Theorem B follow from Lemma 5.20 and Proposition 5.22 respectively. There it is shown that

$$T_{\mathrm{alg}}\pi_* X \cong \bigoplus_{n\geq 0} (\pi_* X)^{\otimes_k n}$$

is a monad on graded k-modules whose algebras are graded associative k-algebras.

Of course graded k-modules and graded associative k-algebras satisfy condition (c). In the category of associative algebras over π_*Y, the abelian group objects are the square-zero extensions of π_*Y such as $\pi_*Y^{S^t} \cong \pi_*Y \oplus \pi_*\Sigma^{-t}Y$ and hence condition (d) from Theorem B is satisfied.

So we obtain a spectral sequence

$$E_2^{s,t} \Longrightarrow \pi_{t-s} A_\infty Hk\text{-}\mathcal{A}lg^d(X,Y)$$

such that

$$E_2^{0,0} = k\text{-}\mathcal{A}lg(\pi_*X, \pi_*Y)$$

and

$$E_2^{s,t} = H^s_{Q,\pi_*Y}(\pi_*X, \pi_*Y^{S^t}) \quad \text{for } t > 0,$$

where the cohomology groups are calculated in the category of graded associative k-algebras over π_*Y. For $s = 0$ these can be identified with the derivations of π_*X into $\pi_{*+t}Y$ [44, §18] and for $s > 0$ these are the Hochschild cohomology groups

$$HH^{s+1}(\pi_*X; \pi_{*+t}Y) \cong \mathrm{Ext}^{s+1}_{\pi_*X\otimes_k(\pi_*X)^{\mathrm{op}}}(\pi_*X, \pi_{*+t}Y)$$

of π_*X with coefficients in $\pi_{*+t}Y$ [41, Prop. 3.6]. Here $\pi_{*+t}Y$ obtains a $\pi_*X \otimes_k (\pi_*X)^{\mathrm{op}}$-module structure by pulling back the $\pi_*Y \otimes_k (\pi_*Y)^{\mathrm{op}}$-module structure along a fixed algebra map coming from $E_2^{0,0}$.

Example 5.4. In the category of Hk-modules, consider the A_∞ algebra $Hk \wedge \Sigma^\infty_+ \Omega SU(n+1)$ for each $n \geq 1$. The homotopy of this algebra is a polynomial algebra $R = R_n$ on generators $\{x_i\}_{1\leq i\leq n}$ with $|x_i| = 2i$. To compute the A_∞ self-maps we take the augmentation map as a base point and apply the T-algebra spectral sequence. By the discussion above we obtain:

$$E_2^{0,0} \cong k\text{-}\mathcal{A}lg(R,R) \cong \prod_{1\leq i\leq n} (R)_{2i}$$
$$E_2^{0,t} \cong \mathrm{Der}(R;\Sigma^{-t}R) \cong \prod_{1\leq i\leq n} (R)_{2i+t}$$
$$E_2^{s,t} \cong \mathrm{Ext}^{s+1}_{R\otimes_k R^{\mathrm{op}}}(R,\Sigma^{-t}R) \quad \text{for } s > 0.$$

In particular, these groups are zero for t odd, hence $E_{2i} = E_{2i+1}$. The Hochschild cohomology groups can be calculated by first pulling back the $R \otimes_k R^{\mathrm{op}} = R \otimes_k R$ action to a different $R \otimes_k R$ action via the isomorphism defined by

$$x_i \otimes 1 \mapsto x_i \otimes 1, \quad 1 \otimes x_i \mapsto x_i \otimes 1 - 1 \otimes x_i.$$

Since $1 \otimes R \subset R \otimes R$ acts trivially on the source we obtain an $(R \otimes_k R)$-free resolution of R by applying $R \otimes_k -$ to the Koszul resolution of k:

$$(\Lambda_k[\sigma x_1,\cdots,\sigma x_n] \otimes_k R \to k) \overset{R\otimes_k -}{\mapsto} (R \otimes_k \Lambda_k[\sigma x_1,\cdots,\sigma x_n] \otimes_k R \to R)$$

Here σx_i has bidegree $(1,2i)$ and $d(\sigma x_i) = 1 \otimes x_i$. Using this resolution and the fact that $1 \otimes x_i$ acts by 0 on the target we see that

$$\mathrm{Ext}^*_{R\otimes_k R^{\mathrm{op}}}(R,R) \cong (\Lambda_k[\sigma x_1,\cdots,\sigma x_n])^* \otimes_k R.$$

So the Hochschild cohomology groups vanish above cohomological degree n and hence the T-algebra spectral sequence is concentrated on the first $n-1$ rows and must collapse at E_n for $n \geq 2$. In particular, if $n = 1$ then the spectral sequence collapses at E_2 onto the 0-line.

Using the vanishing results in the spectral sequence above, we see that there are no obstructions to lifting an arbitrary map of k-algebras

$$H_*\Omega SU(n+1) \to H_*\Omega SU(n+1)$$

to a map of A_∞ algebras if $n \leq 3$ and such a map is unique up to homotopy if $n \leq 2$. For $n = 1$ this result is expected since $\Omega SU(2) \cong \Omega\Sigma S^2$ is stably a free A_∞ algebra.

The previous computation did not depend on the A_∞ algebra $Hk \wedge \Sigma^\infty_+ \Omega SU(n+1)$ so much as the fact that its ring of homotopy groups is polynomial on generators in even degrees. In particular, there are no obstructions to lifting an abstract algebraic isomorphism to an equivalence if there are sufficiently few generators:

Proposition 5.5. Let R_n be a polynomial algebra on n generators in even degrees. Then for $n \leq 3$, there is a unique Hk-algebra V up to homotopy such that $\pi_* V \cong R_n$. In particular, all such algebras are weakly equivalent to the commutative Hk-algebra HR_n.

E_∞-algebras in rational Hk-modules

For the remainder of this section we will study E_∞-algebra maps in the category of Hk-modules for some field k of characteristic 0. To identify the E_2 term we will again apply Theorem B. We first check the hypotheses. Conditions (a) and (b) follow from Lemma 5.20 and Proposition 5.24 respectively. The verification of condition (c) is immediate from the identification of T_{alg} in Proposition 5.24 as the free graded commutative k-algebra monad. The verification of condition (d) proceeds exactly as in the A_∞-case above and the associated Quillen cohomology groups are the classical André-Quillen cohomology groups for graded commutative k-algebras.

Example 5.6. If we allow n to go to infinity in Example 5.4 then $\Omega SU \simeq BU$ is an infinite loop space and consequently $Hk \wedge \Sigma^\infty_+ \Omega SU$ is an E_∞ algebra in Hk-modules.

We have the following identification of the E_2-term, where

$$R = H_*(\Omega SU) \cong k[x_i]_{i\geq 1}$$

and k-$\mathcal{CAlg}$ is the category of commutative k-algebras:

$$\begin{aligned}
E_2^{0,0} &\cong k\text{-}\mathcal{CAlg}(R,R) \cong \prod_{i\geq 1}(R)_{2i} \\
E_2^{0,t} &\cong \mathrm{Der}(R;\Sigma^{-t}R) \cong \prod_{i\geq 1}(R)_{2i+t} \\
E_2^{s,t} &\cong H^s_{AQ,k}(R;\Sigma^{-t}R) \quad \text{for } t>0.
\end{aligned}$$

Since R is a polynomial algebra over k, it is smooth and by Proposition 5.11 all higher André-Quillen cohomology groups vanish. As a consequence we see the spectral sequence collapses at E_2 onto the 0-line. Hence every map of homology rings lifts to a homotopically unique map of E_∞ algebras in Hk-modules.

In general, free algebras have collapsing spectral sequences:

Example 5.7. If $X = TM$ is a free E_∞ ring spectrum then the unit map $X \to TX$ is a map of E_∞ ring spectra and defines a section of the bar resolution. Consequently the T-algebra spectral sequence computing the homotopy of $E^d_\infty(X,Y)$ collapses at E_2 onto the 0-line. So in this case the edge homomorphism $\pi_0 E^d_\infty(X,Y) \to H_\infty(X,Y)$ is an isomorphism. Moreover there is a homotopy equivalence $E^d_\infty(X,Y) \simeq \mathcal{S}pectra^d(M,Y)$.

Remark 5.8. Since rational localization is smashing, the extension functor from E_∞ algebras to E_∞ algebras in $H\mathbb{Q}$ modules is an equivalence for every rational E_∞ ring spectrum. From this we obtain for any rational E_∞ ring spectra X and Y

$$E^d_\infty(X,Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{Alg}^d(H\mathbb{Q}\wedge X, Y) \simeq E_\infty H\mathbb{Q}\text{-}\mathcal{Alg}^d(X,Y).$$

So there is no difference homotopically between the space of E_∞ maps between two rational E_∞ rings and the space of E_∞ algebra maps in $H\mathbb{Q}$-modules.

Example 5.9. We will now construct infinitely many homotopically distinct E_∞ maps that all induce the same H_∞ map. For a space X, recall that the cotensor $H\mathbb{Q}^X$ is an E_∞ ring spectrum satisfying $\pi_* H\mathbb{Q}^X \cong H^{-*}(X;\mathbb{Q})$. Now to calculate the homotopy groups of $E^d_\infty(H\mathbb{Q}^{S^2}, H\mathbb{Q}^{S^3})$ we apply the T-algebra spectral sequence and use the identification of the E_2-term above. As a base point we will take a 'trivial' map ϵ of E_∞ rings induced by a map of the form $S^3 \to * \to S^2$.

Let $\mathrm{Ind}_* A$ denote the graded module of indecomposables of an augmented graded commutative algebra A. To calculate the E_2 term we have

$$E_2^{0,0} \cong \mathbb{Q}\text{-}\mathcal{CAlg}\left(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3}\right) \cong \mathrm{Ind}_{-3}\left(\pi_* H\mathbb{Q}^{S^2}\right) = 0 = \{\epsilon\}.$$

For $t > 0$ we use the map ϵ to regard $\pi_* H\mathbb{Q}^{S^2}$ as a commutative algebra over $\pi_* H\mathbb{Q}^{S^3}$ and obtain

$$E_2^{s,t} \cong H^s_{AQ}\left(\pi_* H\mathbb{Q}^{S^2}; \pi_{*+t} H\mathbb{Q}^{S^3}\right) \cong ho(s\mathbb{Q}\text{-}\mathcal{CAlg}_{\downarrow \pi_* H\mathbb{Q}^{S^3}})\left(\pi_* H\mathbb{Q}^{S^2}, \pi_* H\mathbb{Q}^{S^3} \oplus \Sigma^s_{\mathbb{Q}} \pi_{*+t} H\mathbb{Q}^{S^3}\right)$$

where the right-hand side is the derived homomorphisms of simplicial graded commutative $\mathbb{Q}$-algebras over the simplicially constant graded algebra $\pi_* H\mathbb{Q}^{S^3}$ into the square-zero extension of this algebra by the sth suspension of $\pi_{*+t} H\mathbb{Q}^{S^3}$ in simplicial $\pi_* H\mathbb{Q}^{S^3}$-modules. To simplify the notation, we will present the slightly more general calculation of the André-Quillen cohomology of an exterior $\mathbb{Q}$-algebra on an even-degree generator and for arbitrary coefficients. To calculate these derived homomorphisms we must first construct a cofibrant replacement of the source.

The cofibrant replacement of an exterior algebra (viewed as a constant simplicial object) on an element in degree $2n$ can be constructed as a two stage cell complex: We take the free algebra $\mathbb{Q}[e_{2n}]$ on an element of degree $2n$ and then cone off (in simplicial graded commutative $\mathbb{Q}$-algebras) e_{2n}^2 via an element f_{4n}. In other words, A is defined by the following homotopy pushout diagram in simplicial graded commutative $\mathbb{Q}$-algebras:

$$\begin{tikzcd}
\mathbb{Q}[f_{4n}] \arrow[r, "f_{4n} \mapsto e_{2n}^2"] \arrow[d] & \mathbb{Q}[e_{2n}] \arrow[d] \\
\mathbb{Q}[\mathrm{Cone} f_{4n}] \simeq \mathbb{Q} \arrow[r] & A
\end{tikzcd}$$

Each simplicial homotopy group $\pi_s A$ of this two-stage complex is a graded $\mathbb{Q}$-module. Since homotopy pushouts of simplicial commutative algebras are derived tensor products we find

$$\pi_s A \cong \mathrm{Tor}_s^{\mathbb{Q}[f_{4n}]}(\mathbb{Q}, \mathbb{Q}[e_{2n}]))) \cong \begin{cases} \mathbb{Q}[e_{2n}]/(e_{2n}^2) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By construction A is a cofibrant replacement of our exterior algebra. On further inspection we can see this is a cofibrant replacement in the category of simplical graded commutative algebras augmented over $\mathbb{Q}$.

To calculate the André-Quillen cohomology of a commutative algebra over B we first construct a cofibrant replacement A as above and regard it as a commutative algebra over B via the given map. For example, in our case we have

$$A \xrightarrow{\simeq} \mathbb{Q}[e_{2n}]/(e_{2n}^2) \xrightarrow{\epsilon} B.$$

If M is a B-module we can construct the square-zero extension $B \oplus \Sigma^s M$ whose simplicial homotopy groups are concentrated in degrees 0 and s. By applying the Quillen left adjoint $B \otimes -$ to the domain algebra we obtain an isomorphism:

$$s\mathbb{Q}\text{-}\mathcal{CAlg}_{\downarrow B}(A, B \oplus \Sigma^s M) \cong s\mathcal{Aug}\text{-}B\text{-}\mathcal{CAlg}(B \otimes A, B \oplus \Sigma^s M).$$

There is an equivalence of categories between augmented commutative algebras over B and non-unital commutative B-algebras. This equivalence sends an algebra $B \to C \xrightarrow{f} B$ to $I_B(C) = \ker f$.

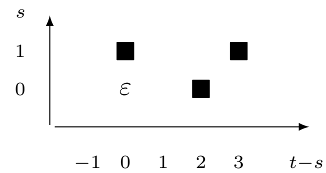

Figure 5.10: T-algebra spectral sequence for E_∞ maps $H\mathbb{Q}^{S^2} \to H\mathbb{Q}^{S^3}$.

Maps into the square-zero extension $B \oplus \Sigma^s M$ in augmented commutative B-algebras correspond to B-module maps from $I_B(C)/I_B(C)^{\otimes_B 2}$ to $\Sigma^s M$. In our case, since the augmentation $A \to B$ factors through an augmentation to $\mathbb{Q}$, and every $\mathbb{Q}$-module is flat, we have

$$I_B(B \otimes A)/\left(I_B(B \otimes A)^{\otimes_B 2}\right) \cong B \otimes \left(I_{\mathbb{Q}}(A)/I_{\mathbb{Q}}(A)^{\otimes 2}\right) = B \otimes \operatorname{Ind}(A).$$

Here we are taking indecomposables levelwise to obtain a simplicial B-module $B \otimes \operatorname{Ind}(A)$. Putting this together we see that

$$s\mathbb{Q}\text{-}\mathcal{CAlg}_{\downarrow B}(A, B \oplus \Sigma^s M) \cong sB\text{-}\mathcal{Mod}(B \otimes \operatorname{Ind}(A), \Sigma^s M) \cong s\mathcal{Mod}_{\mathbb{Q}}(\operatorname{Ind}(A), \Sigma^s(M)).$$

Since the indecomposables functor is a Quillen left adjoint by Proposition 3.5, it takes the cellular pushout diagram of simplicial augmented graded commutative $\mathbb{Q}$-algebras defining A to a homotopy pushout diagram of simplicial graded commutative $\mathbb{Q}$-modules:

$$\begin{tikzcd}
\mathbb{Q}\{f_{4n}\} \arrow[r, "0"] \arrow[d] & \mathbb{Q}\{e_{2n}\} \arrow[d] \\
\operatorname{Ind}(\mathbb{Q}[\operatorname{Cone} f_{4n}]) \simeq 0 \arrow[r] & \operatorname{Ind}(A) \simeq \mathbb{Q}\{e_{2n}\} \oplus \Sigma^1 \mathbb{Q}\{f_{4n}\}
\end{tikzcd}$$

If we let M_k denote the degree k portion of M we see that

$$ho(s\mathcal{Mod}_{\mathbb{Q}})(\operatorname{Ind}(A), \Sigma^s M) = \begin{cases} M_{2n} & \text{if } s = 0, \\ M_{4n} & \text{if } s = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In our case, $n = -1$ and $M = \pi_{*+t} H\mathbb{Q}^{S^3}$, so $ho(s\mathcal{Mod}_{\mathbb{Q}})(\operatorname{Ind}(A), \Sigma^s M)$ consists of a copy of $\mathbb{Q}$ when $(s,t) \in \{(0,-1),(0,2),(1,1),(1,4)\}$ and is zero otherwise. Ignoring the non-contributing term in bidegree $(0,-1)$, we obtain the E_2-term in Figure 5.10.

All other entries are trivial so the spectral sequence collapses at E_2. The $\mathbb{Q}$ in $E_2^{1,1}$ detects an infinite family of homotopically distinct E_∞ maps which, because they land in positive filtration, induce the same H_∞ map ϵ. It can be shown that this infinite family is generated by the morphism of E_∞ rings induced by the Hopf map $S^3 \to S^2$ [39, § 3.2].

In the previous example, the spectral sequence vanished above the 1-line. This guaranteed the collapse of the spectral sequence and provided an algebraic description of the homotopy groups of the space of E_∞ maps. This is because the map $\mathbb{Q} \to \pi_* H\mathbb{Q}^{S^2}$ is a local complete intersection morphism and hence the higher André-Quillen cohomology groups vanish. We will say a morphism $A \to B$ of graded commutative rings is formally a local complete intersection, resp. smooth, resp. étale, if the relative cotangent complex $L_{B/A}$ [41] has projective dimension at most one, resp. is projective, resp. is contractible.

Proposition 5.11. Suppose $f : k \to R$ and $k \to S$ are morphisms of rational E_∞ rings. Suppose the E_2 term of the T-algebra spectral sequence computing the space of E_∞ ring maps under k between R and S can be identified with the André-Quillen cohomology of $\pi_* R$ in $\pi_* k$-algebras as above. Then if the morphism f on homotopy groups is

1. *formally a local complete intersection* then the spectral sequence collapses at the E_2 page onto the 0 and 1 lines and every H_∞ map in k-modules can be realized by an E_∞ map, although possibly non-uniquely.
2. *formally smooth* then the spectral sequence collapses at the E_2 page onto the 0-line and every H_∞ map in k-modules can be realized, uniquely up to homotopy, by an E_∞ map.
3. *formally étale* then the spectral sequence collapses at the E_2 page and $E_2^{s,t} = 0$ if $t > 0$. As a consequence, the mapping space is homotopically discrete and every H_∞ map in k-modules can be realized, up to a contractible space of choices, by an E_∞ map.

Proof. All of the results follow from the vanishing of the relevant André-Quillen cohomology groups [41, Thm. 2.4 (*ii*)] and our identification of $E_2^{0,0}$ with the set of H_∞ maps. □

We remark that in the previous proposition one can make the necessary E_2 identification when $\pi_* R$ is a free $\pi_* k$-module.

Example 5.12. We now construct examples of H_∞ ring maps that do not lift to E_∞ ring maps. The argument below does not make explicit use of the spectral sequence beyond the identification of the H_∞ maps, although it does have consequences for its behavior.

Let M be the Heisenberg 3-manifold: the quotient of the group of uni-upper triangular 3×3 real matrices by the subgroup with all integer entries. Since M is a quotient of a contractible group by a discrete subgroup it is a $K(\pi, 1)$. The commutator subgroup of π is free abelian of rank one and π fits into the short exact sequence of groups

$$1 \to \mathbb{Z} \to \pi \to \mathbb{Z} \times \mathbb{Z} \to 1.$$

In particular M is a nilpotent space.

Applying the classifying space functor to the above exact sequence we see that up to homotopy, M can also be realized as the total space of an S^1 bundle over the torus T^2. This S^1 bundle is classified by the generator of $\mathbb{Z} \cong H^2(T^2; \mathbb{Z}) \cong [T^2, BS^1]$.

A computation with the Serre spectral sequence shows $\pi_* H\mathbb{Q}^M$ is generated by exterior classes x and y in degree -1, polynomial classes α and β in degree -2, which satisfy

$$0 = xy = \alpha^2 = \beta^2 = \alpha\beta = x\alpha = y\beta = x\beta - y\alpha.$$

As a consequence we see:

$$H_\infty(H\mathbb{Q}^M, H\mathbb{Q}^{S^2}) \cong E_2^{0,0} \cong \mathrm{Ind}_{-2}(\pi_* H\mathbb{Q}^M) = \mathbb{Q}\{\alpha, \beta\}. \tag{5.13}$$

There are also Massey product identifications $\alpha \in \langle x, x, y \rangle$ and $\beta \in \langle y, y, x \rangle$ with indeterminacy

$$0 = xH^1(M; \mathbb{Q}) + H^1(M; \mathbb{Q})y.$$

Any map from $H\mathbb{Q}^M$ to $H\mathbb{Q}^{S^2}$ sends x and y to zero for degree reasons. Now α and β are Massey products in x and y and Massey products in $H\mathbb{Q}^*M$ correspond to Toda brackets in $H\mathbb{Q}^M$. Since E_∞ maps preserve Toda brackets, they must also send α and β to zero. So α and β must support differentials and correspond to H_∞ maps which do not lift to E_∞ maps.

The behavior of this spectral sequence is explained further in [39], where it is shown that this spectral sequence is converging to

$$\pi_* \mathcal{T}op^d(S^2, K(\pi,1)_{\mathbb{Q}}) \cong \pi_*(\Omega^2 K(\pi,1) \times K(\pi,1)_{\mathbb{Q}}) \cong \pi_* K(\pi,1)_{\mathbb{Q}}.$$

In particular all non-trivial elements in the $t-s=0$ column, including α and β, must support a differential.

5.3. Coker J *and maps of E_∞ ring spectra*

The following example is a joint result of the second named author and Nick Kuhn.

For this example we will need to recall the definitions of some classical infinite loop spaces and their associated connective spectra [24, p.271]. Let $SL_1S^0 = GL_1S^0\langle 0\rangle$ denote the 1-component of QS^0. Let us fix an odd prime p and let q be an integer generating $(\mathbb{Z}/p^2)^\times$. Define J to be the fiber of the map

$$BU^\otimes \xrightarrow{\psi^q/\mathrm{Id}} BU^\otimes$$

where $BU^\otimes$ is the 1-component of p-local K-theory and ψ^q is the qth Adams operation. The d-invariant defines a map $S^0 \to KU$ which restricts to a map $SL_1S^0 \xrightarrow{D} BU^\otimes$ which in turn lifts to a map $SL_1S^0 \xrightarrow{D} J$. Let Coker J be the fiber of this last map.

At the prime 2 there are several possible definitions of J and consequently several possible definitions of Coker J. A perfectly reasonable approach is to set J to be the fiber of the map

$$BO^\otimes \xrightarrow{\psi^3/\mathrm{Id}} BO^\otimes. \tag{5.14}$$

However this introduces some homotopy groups in low degrees that are not in the image of D. To rectify this there are variations where one replaces one or both copies of BO by either its 1- or 2-connected cover. Rather than go through all the variations we note that all possible choices will yield the same definition of J after taking 1-connected covers. So we define J to be the 1-connected cover of the fiber of the map in (5.14). We then set Coker J to be the fiber of the map $SL_1S^0\langle 1\rangle \xrightarrow{D} J$ defined as above.

It is a non-trivial classical result that all spaces and maps in sight are infinite loop maps (see [24] for a survey of these results), but this can be easily deduced from a more modern construction: For each prime p the above J-space is the simply connected cover of the unit component of the 0th space of the $K(1)$-local sphere. The latter spectrum fits into the analogous fiber sequence with K-theory spectra and the D-invariants are induced by the unit map from the sphere spectrum. It is clear from this construction that all maps are infinite loop maps. We will follow tradition and denote the connective spectra associated to these infinite loop spaces with lower case letters.

Example 5.15. Let $\Sigma^\infty_+ \mathrm{Coker}\, J$ be the unreduced suspension spectrum of the infinite loop space Coker J and let R be any E_∞ ring spectrum. We will show that the T-algebra spectral sequence computing the homotopy of $E^d_\infty(\Sigma^\infty_+ \mathrm{Coker}\, J, L_{K(2)}R)$ collapses at the E_2 page onto the 0-line. So in this case the edge homomorphism

$$\pi_0 E^d_\infty(\Sigma^\infty_+ \mathrm{Coker}\, J, L_{K(2)}R) \to H_\infty(\Sigma^\infty_+ \mathrm{Coker}\, J, L_{K(2)}R)$$

is an isomorphism. Moreover there is a homotopy equivalence of spaces

$$E^d_\infty(\Sigma^\infty_+ \operatorname{Coker} J, L_{K(2)}R) \simeq \Omega^\infty L_{K(2)}R.$$

This will follow from the canonical equivalence

$$E^d_\infty(\Sigma^\infty_+ \operatorname{Coker} J, L_{K(2)}R) \simeq E^d_\infty(L_{K(2)}(\Sigma^\infty_+ \operatorname{Coker} J), L_{K(2)}R),$$

the following result, and Example 5.7.

Theorem 5.16. [Kuhn-Noel] There is a $K(2)$-equivalence of E_∞ ring spectra

$$TS^0 \simeq \Sigma^\infty_+ \operatorname{Coker} J$$

where T is the above monad whose algebras are E_∞ ring spectra.

Proof. A consequence of [31, Thm. 2.21] is that for any spectrum X there is a natural map of E_∞ ring spectra

$$TX \to L_{K(2)}\Sigma^\infty_+\Omega^\infty X \tag{5.17}$$

which is an equivalence if X is 2-connected, $\pi_3 X$ is torsion, and $K(1)_*\Omega^\infty X$ is trivial.

First we consider the p-local case for an odd prime p. In this case the D-invariant

$$SL_1S^0 \xrightarrow{D} J$$

is at least $2p-3$ connected, hence $\operatorname{Coker} J$ is at least 3-connected. As shown in [24, Thm. 2.7], $K(1)_* \operatorname{Coker} J$ is trivial for every prime p. Hence (5.17) is an equivalence for $X = \operatorname{coker} j$.

Delooping the defining fibration for $\operatorname{Coker} J$ we obtain a fiber sequence

$$\operatorname{coker} j \to sl_1S^0 \xrightarrow{d} j.$$

Since j is $K(2)$-acyclic we have a $K(2)$-equivalence $\operatorname{coker} j \to sl_1S^0$. The additive H-space structure on QS^0 induces a homotopy equivalence $SL_1S^0 \simeq QS^0_0$ between the 1 and 0 components of QS^0. Although this is not a map of infinite loop spaces, applying the Bousfield-Kuhn functor $\phi_2 \colon ho\mathcal{T}op_* \to ho(L_{K(2)}\mathcal{S}pectra)$ (which factorizes the $K(2)$-localization functor as $L_{K(2)} \simeq \phi_2 \circ \Omega^\infty$ [30, Thm. 1.1]) to this equivalence does yield an equivalence $L_{K(2)}sl_1S^0 \simeq L_{K(2)}S^0\langle 0\rangle$.

Since Eilenberg-MacLane spectra are $K(n)$-acyclic [42], the defining cofiber sequence for the 0-connected cover

$$S^0\langle 0\rangle \to S^0 \to H\mathbb{Z}$$

shows that $L_{K(2)}S^0\langle 0\rangle \simeq L_{K(2)}S^0$. Finally we use naturality of the spectral sequence

$$H_*(\Sigma_n; K(2)_*(X)^{\otimes_{K(2)_*} n}) \Longrightarrow K(2)_*((E\Sigma_n)_+ \wedge_{\Sigma_n} X^n)$$

to see that the functor T preserves $K(2)$-equivalences.

Assembling these results, we obtain the desired zig-zag of equivalences of E_∞ ring spectra in the $K(2)$-local category

$$TS^0 \leftarrow T(S^0\langle 0\rangle) \leftarrow Tsl_1S^0 \leftarrow T\operatorname{coker} j \to \Sigma^\infty_+ \operatorname{Coker} J.$$

At the prime 2 our defining fibration sequence is

$$\operatorname{Coker} J \to SL_1 S^0\langle 1\rangle \xrightarrow{D} J.$$

Here D is 3-connected so $\operatorname{Coker} J$ is sufficiently connected. Again [24, Thm. 2.7] shows that $K(1)_* \operatorname{Coker} J$ is trivial. The rest of the argument proceeds as before to obtain a zig-zag of $K(2)$-local equivalences of E_∞ ring spectra

$$TS^0 \leftarrow T(S^0\langle 1\rangle) \leftarrow T(sl_1 S^0\langle 1\rangle) \leftarrow T \operatorname{coker} j \to \Sigma^\infty_+ \operatorname{Coker} J.$$

□

Remark 5.18. Combining this result with [28, 49] one can determine the $K(2)$ and E_2-cohomology of $\operatorname{Coker} J$.

Remark 5.19. The $K(1)$-local analogue of Example 5.15 is considered in [39, § 3.3].

5.4. Computational lemmas

One of the key steps in obtaining a calculational description of the E_2 term is verifying conditions (a) and (b) from Theorem B. In our examples, condition (a) follows from the following result in the case $R = Hk$ for some field k:

Lemma 5.20. If M and N are R-modules such that $\pi_* M$ is projective as a $\pi_* R$-module then

$$\pi_t\left(\mathcal{M}od^d_R(M,N)\right) \cong \mathcal{M}od_{\pi_* R}(\pi_* M, \pi_{*+t} N).$$

Proof. The Ext spectral sequence of [18, Thm. IV.4.1] collapses. □

To verify condition (b) from Theorem B we need to find a monad T_{alg} such that there is a natural isomorphism

$$\pi_* T \cong T_{\mathrm{alg}} \pi_*.$$

When T is the monad associated to A_∞/E_∞-algebras in Hk-modules for suitable k we will identify T_{alg}. The associated categories of T_{alg}-algebras will be equivalent to graded associative/commutative k-algebras respectively.

In both of these examples our monad takes the form

$$TM = \bigvee_{n\geq 0} K_n \otimes_{\Sigma_n} M^{\wedge_{Hk} n}.$$

In the E_∞ case K_n is a contractible free Σ_n-space while in the A_∞ case it is a free Σ_n-space which is weakly equivalent to Σ_n. The identification of $\pi_* TM$ as a functor of $\pi_* M$ in these cases follows from the following sequence of elementary spectral sequence arguments which we have stated in terms of a fixed commutative S-algebra R.

Lemma 5.21. If M and N are R-modules such that either $\pi_* M$ or $\pi_* N$ is flat as a $\pi_* R$ module then

$$\pi_*(M \wedge_R N) \cong \pi_* M \otimes_{\pi_* R} \pi_* N$$

Proof. The Tor spectral sequence of [18, Thm. IV.4.1] collapses. □

Proposition 5.22. Suppose that M is an R-module spectrum, π_*R is a graded field, and T is the above monad on R-module spectra whose category of algebras is the category of A_∞ algebras in R-module spectra. Then there is a natural isomorphism

$$\pi_*TM \cong T_{\mathrm{alg}}\pi_*M := \bigoplus_{n\geq 0}(\pi_*M)^{\otimes_{\pi_*R}n}.$$

Here T_{alg} is the monad on π_*R-modules whose algebras are the associative algebras in that category.

For the E_∞ case we will need the following:

Lemma 5.23. If $E\Sigma_n$ is a contractible Σ_n-CW-complex and M is an R-module such that $n!$ is a unit in π_0R then

$$\pi_*(E\Sigma_n \otimes_{\Sigma_n} M^{\wedge_R n}) \cong \pi_*(M^{\wedge_R n})/\Sigma_n.$$

Proof. The homotopy orbit spectral sequence

$$H_s(\Sigma_n;\pi_t(M^{\wedge_R n})) \Longrightarrow \pi_{s+t}((E\Sigma_n)_+ \wedge_{\Sigma_n} M^{\wedge_R n})$$

collapses by a standard transfer argument since $|\Sigma_n| = n!$ acts invertibly on the coefficients. □

Proposition 5.24. Suppose that M is an R-module spectrum, π_*R is a graded field with π_0R a field of characteristic 0, and T is the above monad on R-module spectra whose category of algebras is the category of E_∞ algebras in R-module spectra. Then there is a natural isomorphism

$$\pi_*TM \cong T_{\mathrm{alg}}\pi_*M := \bigoplus_{n\geq 0}(\pi_*M)^{\otimes_{\pi_*R}n}/\Sigma_n.$$

Here T_{alg} is the monad on π_*R-modules whose algebras are the commutative algebras in that category.

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