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SOME APPLICATIONS OF THE THEORY OF FORMAL GROUPS TO ALGEBRAIC TOPOLOGY

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I love deadlines. I love the whooshing sound they make as they fly by.

- Douglas Noel Adams

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ABSTRACT

This dissertation provides several contributions to stable homotopy theory. It is divided into the following three chapters:

Generalized Witt schemes in algebraic topology

We analyze the even-periodic cohomology of the space BU and some of its relatives using the language of formal schemes as developed by Strickland. In particular, we connect $E^0(BU)$ to the theory of Witt vectors and λ-rings. We use these connections to study the effect of the coproduct arising from the tensor product on generalized Chern classes. We then exploit this connection to simultaneously construct Husemoller's splitting of $H\mathbb{Z}^*_{(p)}(BU)$ and Quillen's splitting of $MU_{(p)}$.

H_∞ orientations on BP

In joint work with Niles Johnson, we show, at the primes 2 and 3, that no map from MU to BP defining a universal p-typical formal group law on BP is H_∞. In particular, no such map is E_∞.

This builds on McClure's work on determining if Quillen's orientation on BP is an H^2_∞ map. By direct computation, we show that the necessary condition he derives for Quillen's orientation to be H^2_∞ fails at the primes 2 and 3. We go on to show that this implies the more general result above.

We also provide a reinterpretation of McClure's conditions in the language of formal group laws.

$H_\infty \neq E_\infty$

We give an example of a spectrum with an H_∞ structure which does not rigidify to an E_3 structure. It follows that not every H_∞ ring spectrum comes from an underlying E_∞ ring spectrum. After comparing definitions, we see the counterexample to the transfer conjecture constructed by Kraines and Lada can be used to construct this spectrum.

CHAPTER 1
INTRODUCTION

The work of Quillen [Qui69a] and Adams [Ada74] connecting the theory of formal group laws to stable homotopy theory provides an organizing program for contemporary stable homotopy theory. Through this connection we obtain the chromatic filtration on stable homotopy groups. This filtration provides an important organizational tool, breaks up many formidable calculations into more tractable pieces, and provides the underlying motivation for many new computational methods.

Morava was a strong advocate of using the language of algebraic geometry in the study of algebraic topology. He conjectured that the above connection was quite deep. In many ways, Morava was right; his ideas led others to construct elliptic cohomology theories, topological modular forms, and a theory of derived algebraic geometry.

In this dissertation we analyze two components of this connection:

- The use of algebraic geometry in algebraic topology before the invention of derived algebraic geometry.

  As discussed below in Section 1.1, we will give an expository account of the translation between algebraic topology and the theory of schemes and formal schemes. In particular, we will find topological objects which correspond to twisted forms of the Witt scheme. We then apply this identification to simultaneously construct several classical splittings in algebraic topology and to provide a new formalism for the splitting principle.

- The link between geometric topology and algebraic topology provided by the identification of the complex cobordism ring and the Lazard ring classifying formal group laws.

  As described more explicitly in Section 1.2, we will show that p-locally this identification does not respect all of the structure involved.

1.1 Witt schemes in algebraic topology

Chapter 2 begins with an introduction to the theory of affine schemes and formal schemes suitable for an algebraic topologist and can be used as a gentle introduction to the more complete theory developed in [Str00b]. Restricting to the affine case allows us to use the Yoneda lemma as a fundamental tool for switching between a representing algebra and its associated affine scheme. We will see that it is frequently easier to understand any additional structure an algebra might carry scheme-theoretically.

Our primary interest is in schemes and formal schemes of the form $\operatorname{Spec} E_0(X)$ and $\operatorname{Spf} E^0(X)$, where E is an even-periodic spectrum and X is a space related to BU. These schemes are rather boring from an algebro-geometric viewpoint; they are usually a form of affine n-space. However, many of the spaces we will consider have product structures which make their associated schemes into group or ring objects.

An affine formal ring scheme is represented by a fairly complex object: a topological algebra with two coproducts that satisfy a distributive law. These coproducts are both topological algebra maps, one of which makes this algebra into a topological Hopf algebra. Similar to the difficulties involved with constructing injective vs projective resolutions, coalgebra structures are usually more difficult to work with than product structures. The anti-equivalence between the categories of (suitably topologized) commutative rings and (formal) affine schemes allows us to translate our coproducts into products.

Our prototypical example of a formal scheme is:

$$\widehat{\mathbb{G}} = \operatorname{Spf}(E^0(\mathbb{C}P^\infty)).$$

The product operation

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty$$

encoding the tensor product operation on line bundles makes $\widehat{\mathbb{G}}$ into a formal group. The formal group structure combined with a specified isomorphism to the formal affine line defines and is defined by the formula in E-theory for the first Chern class of a tensor product of line bundles in terms of the Chern classes of the constituent bundles.

Identifying $\mathbb{C}P^\infty$ with BS^1, the inclusion of a maximal torus into $U(n)$ induces a map of formal schemes

$$\mathrm{Spf}(E^0(\mathbb{C}P^\infty)^{\otimes n}) \to \mathrm{Spf}(E^0(BU(n))). \tag{1.1.1}$$

Both of these schemes can be identified with formal affine n-space $\widehat{\mathbb{A}}^n$. However, standard transfer arguments show that

$$E^0(BU(n)) \cong (E^0(\mathbb{C}P^\infty)^{\otimes n})^{\Sigma_n}$$

where $(\mathbb{C}P^\infty)^n$ is equipped with the obvious permutation action. As a consequence we can use the anti-equivalence between such rings and affine formal schemes to identify $\mathrm{Spf}(E^0(BU(n)))$ as the orbits of formal affine n-space under permuting coordinates.[1]

The author invented the notion of a symmetric scheme to encode the relationship between the above two schemes and to provide simple objects that translate easily into the theory of Λ-schemes and Witt schemes. In the examples above, we can identify $\mathrm{Spf}(E^0(BU(n)))$ and $\mathrm{Spf}(E^0(\mathbb{C}P^\infty)^{\otimes n})$ with the functors whose R-points are monic degree n polynomials which split over R and the ordered n-tuples of (nilpotent) roots of these polynomials respectively. Now the map in Equation 1.1.1 can be identified with the map taking a set of roots η_i to the polynomial

$$\prod_{i=1}^{n}(x - \eta_i).$$

Since Λ-schemes and Witt schemes admit simple descriptions in terms of formal power series, taking a colimit over increasing n makes it is easy to translate between these different schemes.

Using these correspondences we can identify $\mathrm{Spf}(E^0(BU))$ with the Cartier dual of the (big) scheme of Witt vectors over E^0. Under this identification we are able to show that Husemoller's splitting of $H\mathbb{Z}^*_{(p)}(BU)$ and Quillen's splitting of $MU_{(p)}$ can be constructed from the splitting of the Witt scheme p-locally. We also show that the formal properties of these schemes encode the splitting principle in algebraic topology. As an application we obtain formulas for the E-theory Chern classes of a tensor product of two vector bundles.

1. The fact that $\mathrm{Spf}(E^0(BU(n)))$ can be identified with affine n-space provides an example of how poorly colimits in affine schemes correspond to our geometric intuition.

1.2 H_∞ ring spectra

Chapter 3 represents joint work with Niles Johnson. In this chapter we provide damning evidence against a well-known and long-standing open conjecture:

Conjecture 1.2.1. *The Brown-Peterson spectrum BP admits an E_∞ ring structure.*

This conjecture is as old as the theory of E_∞ ring spectra. To understand why it is probably false, we first examine why it should be true:

1. The Brown-Peterson spectrum is a summand of the p-local complex cobordism spectrum $MU_{(p)}$, which is equipped with a very natural E_∞ ring structure.

2. The complex cobordism spectrum is *very* well-behaved computationally and most calculations in BP theory are immediately deduced (using the splitting in (1)) from calculations in cobordism.

3. As a consequence of (1), the role that complex cobordism plays in computing stable homotopy groups of spheres (via the Adams-Novikov spectral sequence) is eclipsed by that of BP in all p-local computations. Essentially no information is lost in this substitution and, over time, the Brown-Peterson spectrum has become the more dominant player in stable homotopy theory.

4. Any attempts to disprove the conjecture so far have been unsuccessful. One might chalk this up to a consequence of (1) and (2).

5. In a very crude sense, the Brown-Peterson spectrum is approximated by a series of E_∞ ring spectra (the Morava E_n-theories).

With the exception of (5), which we discount because of the rather large leaps of logic underlying the statement, each of these reasons is dependent on (1). Indeed, if anyone were to prove that BP has an E_∞ ring structure, their first task would be to show that this structure is compatible with the splitting of $MU_{(p)}$.

This splitting is constructed in the stable homotopy category. In fact, every construction of the Brown-Peterson spectrum is homotopical in nature. This is not promising, since nearly every E_∞ ring spectrum is constructed from a nice point-set model. If we try to show that BP does not admit an E_∞ structure compatible with the above splitting, then our hands are tied by not having a given point-set model for BP.

Now Hovey and Strickland have shown that any two models for BP are canonically homotopy-equivalent as spectra [HS99]. We can then ask if we can also lift an E_∞ ring structure in the homotopy category—that is, an H_∞ ring structure—to an honest E_∞ ring structure in the category of spectra.

The theories of E_∞ and H_∞ ring spectra are both quite rich. An H_∞ ring spectrum is equipped with a collection of power operations which can be used to prove many central theorems in homotopy theory. After consulting with several experts in early 2009, it appeared that every known H_∞ ring spectrum arose by taking an E_∞ ring spectrum and passing to the stable homotopy category. One could naively hope that every H_∞ ring spectrum could be realized by an E_∞ ring spectrum.

This claim is the spectrum-level analogue of the transfer conjecture:

Conjecture 1.2.2 (Attributed to Quillen)**.** *If X is a group-like commutative H-space, such that the Yoneda embedding $[-, X]$, of $X \in \mathcal{hTop}_*$ into $\mathcal{AbGrp}^{\mathcal{hTop}_*^{op}}$, admits a system of additive transfer maps (i.e., X is a transfer space) then X is homotopy equivalent to an infinite loop space.*

We connect this conjecture to the above discussion by the following well-known results.

1. Group-like E_∞ spaces correspond to infinite loop spaces up to weak homotopy equivalence [May72].

2. Adding a disjoint basepoint and taking suspension spectra takes additive transfers to multiplicative transfers.

3. Admitting such a system of multiplicative transfers is equivalent (up to $\lim^1$ issues) to having a sequence of power operations that define an H_∞ structure [BMMS86].

In Chapter 4, the author makes the above analogy precise. In Theorem 4.1.2 we use the above results and the counterexample to the transfer conjecture constructed by Kraines and Lada [KL79] to provide an explicit example of a spectrum with an H_∞ structure that does not lift to an E_∞ structure .

We now begin to see why Conjecture 1.2.1 about BP has been so intractable. In order to obtain useful computational information for obstruction theory arguments, we need to make use of the splitting maps in (1). This construction of BP, as well as others, is purely homotopical and *a priori* should only endow BP with an H_∞ structure. Even if we find such a structure, by the above discussion we still have the generally non-trivial task of rigidifying it to an E_∞ structure.

We step up to the first task only to discover a good deal of work has already been done for us. We need to see if either of the splitting maps below could be maps of H_∞ ring spectra.

$$BP \xrightarrow{s} MU_{(p)} \xrightarrow{r} BP$$

Hu, Kriz, and May have already shown that there are no H_∞ ring maps from BP into $MU_{(p)}$ [HKM01]. It follows that the idempotent map

$$\varepsilon = sr : MU_{(p)} \to MU_{(p)}$$

is not an H_∞ ring map and that there is no formal reason for

$$BP \simeq MU_{(p)}[\varepsilon^{-1}]$$

to be an H_∞ ring spectrum.

However, most applications make use of ring maps out of MU, also called orientations for geometric reasons. When r is Quillen's splitting map, Jim McClure has given a necessary and sufficient condition for r to be an H_∞^2 map, which would imply r is an H_∞ map [BMMS86, VII].

In Chapter 3 we take McClure's work and show that any H_∞ map to BP which endows it with those properties listed in (3) must satisfy McClure's condition. By brute force calculation we show that this condition *is not satisfied* when p is 2 or 3 (Theorem 3.2.2). We see no reason for these primes to be special and make the following conjecture:

Conjecture 1.2.3. *There are no H_∞ or E_∞ orientations on the Brown-Peterson spectrum.*

Theorem 3.2.2 negates most of the support for Conjecture 1.2.1. As for (5), the proof of the above result combined with Matt Ando's formulas for the H_∞ orientations on the

Morava E_n-theories show, at the primes 2 and 3, there are no H_∞-orientations of E_n that factor through BP by ring maps. As additional support for the above conjecture, in a separate work Niles Johnson and the author have shown that for *all primes* there are no H_∞ orientations of E_1 that factor through BP.

CHAPTER 2
GENERALIZED WITT SCHEMES IN ALGEBRAIC TOPOLOGY

2.1 Introduction

It is also very tempting to declare that at this date all such results on ordinary homology [of BU and BSU] may be assumed known; and if they are not on record, why, that is a defect in the papers written ten or twenty years ago, and not in the present one. Unfortunately, a sense of duty impels me to sketch a proof.

- J.F. Adams 1976

This paper picks up threads left by Ben-Zvi [BZ95] and Strickland [Str00b] and weaves them together. In Ben-Zvi's minor thesis he observes that the cohomology ring $H^*(BU)$ is the ring of functions on the Cartier dual of the Witt scheme. While Strickland is developing the proper foundations for making these kind of connections in [Str00b] he also remarks that the formal scheme associated to $E^0(\coprod_{n\geq 0} BU(n))$, for E an even-periodic cohomology theory, is a graded λ-semiring (or rig) object. Using Strickland's framework to analyze the E cohomology of a space by studying the associated formal scheme $\mathrm{Spf}(E^0(BU))$ and the algebra of Witt schemes described in [BZ95, Haz78], we will flesh out and connect these remarks.

This continues a tradition initiated by Morava in the 70's, a strong advocate of applying the language and tools of algebraic geometry to the study of algebraic topology. Some of his ideas eventually flowered into the field of derived algebraic geometry. In this paper, we will study some of the ordinary algebraic geometry that motivated the creation of this blossoming field.

Witt vectors appear in many places in algebraic topology including the Husemoller-Witt splitting [Hus71], Hodgkin's calculation of the K-theory of QS^0 [Hod72], the classification of bicommutative Hopf algebras [Sch70, Goe98] and in the formulas for formal group laws [Rav00].

Individually the results below are in the literature although they may require some translation since they appear in different contexts. For example, our form of the algebraic splitting principle Theorem 2.3.9, can be constructed from different forms of the splitting principle that appear in the literature. We have made a significant effort to cite relevant sources, but some of the topics discussed below have been studied by many people for quite some time. By juxtaposing the results spread out across the literature we aim to clarify the connections between them.

This paper is written for the general algebraic topologist. We therefore assume familiarity with some category theory, especially representable functors and the Yoneda lemma as well as the theory of vector bundles and classifying spaces laid out in [Hus94, May99]. But we do not assume the reader is familiar with the basics of affine schemes or ind-objects. To keep this paper mostly self-contained, in Section 2.2 we provide a short treatment of these topics referring to [Str00b] for a more in-depth treatment.

First, in Section 2.2.1 we recall the relevant material about schemes, while emphasizing the role of the representing algebra. After reviewing the fundamentals of ind and pro-objects in Section 2.2.2 we then proceed to the theory of formal schemes in Section 2.2.3 and Cartier duality in Section 2.2.4.

In Section 2.3 we introduce the star players for the algebraic geometry team. We introduce our theory of symmetric schemes which provide a simple intermediary between Λ-schemes and Witt schemes. After recalling some aspects of the theory of λ-rings and Witt vectors, including their p-local splittings, (see [Haz78, BZ95] or the recent survey [Haz08]) we explicitly describe the equivalences between them. Symmetric schemes encode the splitting principle rather explicitly, allowing us to easily define operations on symmetric schemes.

In Section 2.4 we introduce the players for the algebraic topology team. We identify the schemes associated to the even-periodic co/homology of spaces related to BU with some of the schemes above and apply our algebraic results. In particular, we use the p-local splitting of the Witt scheme to construct the Husemoller splitting of the cohomology of BU and Quillen's splitting of $MU_{(p)}$ simultaneously. The observation that these splittings can be constructed simultaneously appears to be new.

We also apply this framework to give low-dimensional formulas—which although well-known to some experts, apparently have never been published—for the generalized Chern class of a tensor product of (stable) vector bundles in Section 2.5.

2.1.1 Conventions

In this paper:

- A ring will always be commutative, associative, and unital.
- A ring without identity will be called a rng.
- A ring without negatives (additive inverses) will be called a rig.
- All binary operations considered will be associative and dually all co-operations will be coassociative.
- All schemes and formal schemes will be affine.

2.2 Affine algebraic geometry

2.2.1 Schemes

We are interested in studying the cohomology of suitably nice spaces that admit a homotopy commutative and unital product. The cohomology rings of such spaces come equipped with a cocommutative and counital coproduct. Being more comfortable with product structures we choose to work in the opposite category of rings or more precisely, the category of affine schemes. In addition to easing our study of the comultiplicative structure on these cohomology rings, schemes provide interesting alternative characterizations of these rings.

Recall that the essential image of an embedding $F : \mathcal{C} \to \mathcal{D}$ is the full subcategory of $\mathcal{D}$ whose objects are isomorphic to some object in the image of F.

Definition 2.2.1. *The category of affine schemes, $\mathcal{Sch}$, is defined to be the essential image of the Yoneda embedding:*

$$\begin{aligned} \mathrm{Spec} : \mathcal{Ring}^{op} &\longrightarrow \mathcal{Set}^{\mathcal{Ring}} \\ \mathrm{Spec}(R) : S &\mapsto \mathcal{Ring}(R, S). \end{aligned}$$

The value of a scheme X on a ring S, is called the set of S-points of X which we denote by $X(S) = \mathcal{Sch}(\mathrm{Spec}(S), X)$.

By definition, $\mathcal{Sch}$ is equivalent to $\mathcal{Ring}^{op}$.

Example 2.2.2. The affine line $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{Z}[x])$, takes a ring to its underlying set. In other words, $\mathbb{A}^1$ is isomorphic to the forgetful functor from rings to sets.

Example 2.2.3. The scheme $\mathbb{A}^1 \setminus \{*\} \cong \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$, takes a ring to the set of units in that ring.

Example 2.2.4. The scheme $\mathrm{Nil}_n \cong \mathrm{Spec}(\mathbb{Z}[x]/(x^n))$, takes a ring R to the set of x in R, such that $x^n = 0$.

To discuss various algebraic categories in $\mathcal{Sch}$, such as group schemes, we will need finite products. The product in schemes arises from the tensor product in rings. As an example we consider affine n-space

$$\begin{aligned} \mathbb{A}^n &\cong \left(\mathbb{A}^1\right)^{\times n} \\ &= \mathrm{Spec}\,\mathbb{Z}[t_1] \times \cdots \times \mathrm{Spec}\,\mathbb{Z}[t_n] \\ &\cong \mathrm{Spec}(\mathbb{Z}[t_1] \otimes \cdots \otimes \mathbb{Z}[t_n]) \\ &\cong \mathrm{Spec}\,\mathbb{Z}[t_1, \ldots, t_n]. \end{aligned}$$

Remark 2.2.5. In fact, $\mathcal{Sch}$ is complete and cocomplete because $\mathcal{Ring}$ is complete and cocomplete, however we must comment that the colimits in affine schemes constructed using this equivalence do not generally agree with those of the larger categories of non-affine schemes or $\mathcal{Set}^{\mathcal{Ring}}$. Since the reader might have little intuition for schemes, we will try to emphasize their role as set-valued functors and clarify the differences between these perspectives.

Later we will need to work with k-algebras, for an arbitrary ring k, whose scheme theoretic analogues are schemes over Spec(k). The category of schemes over Spec(k) is equivalent to the essential image of the Yoneda embedding:

$$\mathrm{Spec}_k : k\text{-}\mathcal{alg}^{op} \longrightarrow \mathcal{Set}^{k\text{-}\mathcal{alg}},$$

where k-$\mathcal{alg}$ is the category of k-algebras (i.e., rings under k). We denote the category of schemes over Spec(k) by $\mathcal{Sch}_k$. Note that the isomorphism of categories $\mathcal{Sch} \cong \mathcal{Sch}_{\mathbb{Z}}$, shows that the relative theory is more general.

Remark 2.2.6. In our examples we have elected to define our schemes over $\mathbb{Z}$, but we could just as easily define their analogues over an arbitrary base ring. Rather than clutter the notation we have elected to leave it to the reader to replace the integers with her preferred base whenever she sees fit.

Tensor products over k correspond to products in $\mathcal{Sch}_k$ and they agree with those in $\mathcal{Set}^{k\text{-}\mathcal{alg}}$. Using this product structure we can construct the category of group schemes over Spec k, $\mathcal{GrpSch}_k$. We should remark that group schemes are the dual of a more familiar concept:

Proposition 2.2.7. *The categories of bicommutative Hopf algebras over k and group schemes over k are antiequivalent.*

Example 2.2.8. The ring $\mathbb{Z}[x]$ with augmentation $\epsilon_+ : \mathbb{Z}[x] \to \mathbb{Z}$ determined by $\epsilon_+(x) = 0$ is an augmented $\mathbb{Z}$-algebra. The maps

$$\begin{array}{rclrcl} \epsilon_+ : \mathbb{Z}[x] & \to & \mathbb{Z} & x & \mapsto & 0 \\ \Delta_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x_1, x_2] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\ \chi_+ : \mathbb{Z}[x] & \to & \mathbb{Z}[x] & x & \mapsto & -x \end{array}$$

make $\mathbb{Z}[x]$ into a cocommutative cogroup (i.e., a bicommutative Hopf algebra). Applying Spec to $\mathbb{Z}[x]$ and the above maps defines the additive group scheme $\mathbb{G}_a$. We can identify $\mathbb{G}_a$ with the forgetful functor from rings to abelian groups.

Example 2.2.9. The ring

$$\mathbb{Z}[x^{\pm}] \equiv \mathbb{Z}[x, x^{-1}]$$

with augmentation $\epsilon_\times(x) = 1$ can be made into a cocommutative cogroup using the maps

$$\begin{array}{rclcrcl} \epsilon_\times : \mathbb{Z}[x^\pm] & \to & \mathbb{Z} & & x & \mapsto & 1 \\ \Delta_\times : \mathbb{Z}[x^\pm] & \to & \mathbb{Z}[x_1^\pm, x_2^\pm] & & x & \mapsto & x_1 \otimes x_2 \\ \chi_\times : \mathbb{Z}[x^\pm] & \to & \mathbb{Z}[x^\pm] & & x & \mapsto & x^{-1}. \end{array}$$

The corresponding group scheme $\mathbb{G}_m$ is called the multiplicative group scheme since it takes a ring to its group of units.

Remark 2.2.10. As an aside, we note that $\mathbb{G}_m$ plays a role in extending the antiequivalence between the categories of rings and affine schemes to graded rings. There is an equivalence between the category of graded algebras and the category of comodules over the Hopf algebra representing $\mathbb{G}_m$ [Str00b, 2.96]. Applying Spec to everything in sight we end up with an antiequivalence between the category of graded algebras and the $\mathbb{G}_m$-equivariant category of affine schemes.

We can combine some of the structure in Example 2.2.9 with that of Example 2.2.8 to define the identity ring scheme.

Example 2.2.11. The ring $\mathbb{Z}[x]$ with augmentations ϵ_+ and $\epsilon_\times$ and comultiplications Δ_+ and $\Delta_\times$ equipped with the coinverse map χ_+ make $\mathbb{Z}[x]$ into a coring. By applying Spec we obtain the ring scheme Id, which takes a ring to itself.

2.2.2 Ind-objects and pro-objects

Before proceeding to the theory of formal schemes, we need to recall some standard facts about ind/pro objects. We encourage the reader to consult [Joh82, Gro64] for a more thorough treatment of this material.

Definition 2.2.12. *A small category $\mathcal{D}$ is called* cofiltered *if*

1. *$\mathcal{D}$ is non-empty.*
2. *For every $X, Y \in \mathcal{D}$, there exists an object $Z \in \mathcal{D}$ and morphisms $f : Z \to X$ and $g : Z \to Y$.*

3. *For every two arrows $f, g : X \to Y$ there exists an object $Z \in \mathcal{D}$ and a morphism $h : Z \to X$ such that $fh = gh$.*

It follows immediately from the definition that any product of cofiltered categories is cofiltered.

Definition 2.2.13. *Given a category $\mathcal{C}$ with small hom-sets and a functor $F : \mathcal{D} \to \mathcal{C}$ from a cofiltered category $\mathcal{D}$, we define the ind-object "colim" $F \in \mathcal{S}et^{\mathcal{C}^{op}}$ by*

$$\text{“colim” } F \in \mathcal{S}et^{\mathcal{C}^{op}} \equiv \operatorname*{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i)),$$

where the colimit is computed in $\mathcal{S}et^{\mathcal{C}^{op}}$.

We compute the morphisms between two ind-objects to be

$$\begin{aligned} \mathcal{S}et^{\mathcal{C}^{op}}(\text{“colim” } F, \text{“colim” } G) &= \mathcal{S}et^{\mathcal{C}^{op}}(\operatorname*{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i)), \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(-, G(j))) \\ &\cong \lim_{i \in \mathcal{D}} \mathcal{S}et^{\mathcal{C}^{op}}(\mathcal{C}(-, F(i)), \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(-, G(j))) \\ &\cong \lim_{i \in \mathcal{D}} \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(F(i), G(j))). \end{aligned}$$

The first isomorphism follows from the definition of a colimit and the second by the Yoneda lemma and that colimits in functor categories are computed pointwise.

It follows that the full subcategory of $\mathcal{S}et^{\mathcal{C}^{op}}$ consisting of objects isomorphic to an ind-object has small hom-sets. This is the category $\operatorname{Ind}\mathcal{C}$ of ind-objects in $\mathcal{C}$.

The functor category $\mathcal{S}et^{\mathcal{C}^{op}}$ is complete because $\mathcal{S}et$ is complete. If $\mathcal{C}$ also has finite products then the product

$$\text{“colim” } F \times \text{“colim” } G \in \mathcal{S}et^{\mathcal{C}^{op}}$$

can be realized as an ind-object:

$$\begin{aligned} \text{“colim” } F \times \text{“colim” } G &= \operatorname*{colim}_{i \in \mathcal{D}} \mathcal{C}(-, F(i)) \times \operatorname*{colim}_{j \in \mathcal{E}} \mathcal{C}(-, G(j)) && (2.2.14) \\ &\cong \operatorname*{colim}_{(i,j) \in \mathcal{D} \times \mathcal{E}} \mathcal{C}(-, F(i) \times G(j)), && (2.2.15) \end{aligned}$$

where the isomorphism follows from cofiltered colimits commuting with finite products in $\mathcal{Set}$[Bor94, 2.13.4]. Using this we can relate the algebraic categories in $\mathcal{C}$ to those in $\operatorname{Ind}\mathcal{C}$.

If $\mathcal{C}$ is equivalent to a category of representable functors then we can think of $\operatorname{Ind}\mathcal{C}$ as formally adjoining cofiltered colimits to $\mathcal{C}$:

Theorem 2.2.16 ([Joh82, Section VI]). *Suppose that $\mathcal{D}$ is the subcategory of functors in $\mathcal{Set}^{\mathcal{C}^{op}}$ such that for any object $X \in Obj(\mathcal{D})$, $X \cong \mathcal{C}(-,Y)$ for some Y. Then* $\operatorname{Ind}\mathcal{D}$ *is equivalent to the subcategory $\mathcal{E}$ of functors in $\mathcal{Set}^{\mathcal{C}^{op}}$, such that for all $\overline{X} \in Obj(\mathcal{E})$, $\overline{X} = \operatorname{colim} Y_i$. where $Y_i \in \mathcal{D}$.*

The definition of a pro-object is precisely dual to that of an ind-object.

Notation 2.2.17. *We denote the category of pro-objects in $\mathcal{C}$ by* $\operatorname{Pro}\mathcal{C}$.

Under the tautological equivalence

$$\operatorname{Pro}\mathcal{C} \simeq \operatorname{Ind}\mathcal{C}^{op}, \tag{2.2.18}$$

we obtain results for pro-objects dual to those above.

2.2.3 Formal schemes

Following [Str00b], we define the category of formal schemes, $\mathcal{FSch}$, as the full subcategory of objects in $\mathcal{Set}^{\mathcal{Ring}}$ which are isomorphic to a cofiltered colimit of affine schemes. This category is equivalent to the category of ind-schemes. By identifying a scheme with a constant ind-scheme we can embed $\mathcal{Sch}$ as a full subcategory of $\mathcal{FSch}$.

Using the equivalence with Ind-schemes, $\mathcal{FSch}$ is equivalent to the opposite category of pro-rings. A pro-ring $R =$ "lim" R_i can be identified with the topological ring $R' = \lim R_i$ where the inverse limit is taken in topological spaces and each R_i represents a discrete topological space [Joh82]. In this description a map of pro-rings corresponds to a continuous map. When the context is clear we will identify such a topological ring with its associated pro-ring.

The equivalence $\operatorname{Pro}\mathcal{Ring}^{op}$ to $\mathcal{FSch}$ is given by the functor

$$\operatorname{Spf} : \text{“lim” } R_i \mapsto \operatorname{colim} \operatorname{Spec}(R_i).$$

Remark 2.2.19. This generalizes the definition of formal schemes in algebraic geometry. An affine formal scheme in that context is one of the form $\mathrm{Spf}(\widehat{R})$, where $\widehat{R} =$ "lim" R/I^n ([Har77, Section II.9]). Each such formal scheme has a geometric interpretation, that does not always hold in our category.

Example 2.2.20. The formal affine line

$$\begin{aligned}\widehat{\mathbb{A}}^1 &= \mathrm{Spf}(\mathbb{Z}[\![x]\!]) \\ &= \operatorname{colim} \mathrm{Spec}(\mathbb{Z}[x]/x^n) \\ &\cong \operatorname{colim} \mathrm{Nil}_n \\ &= \mathrm{Nil},\end{aligned}$$

takes a ring to the set of nilpotent elements of that ring.

Notation 2.2.21. *Analogous to the informal case, we have the category of formal schemes over a given (formal) scheme X, which we denote by $\mathcal{FSch}_X$.*

Since colimits commute with (cofiltered) colimits we see that $\mathcal{FSch}$ is cocomplete and its colimits arise from those in schemes. By Theorem 2.2.16, cofiltered colimits of formal schemes are preserved under the inclusion $\mathcal{FSch} \to \mathcal{Set}^{\mathcal{Ring}}$.

Using Equation 2.2.15 we see that the products in schemes are related to the products in formal schemes. For example, if $X = \operatorname{colim}_{i \in I} X_i$ and $Y = \operatorname{colim}_{j \in J} Y_j$ then

$$X \times Y \cong \operatorname*{colim}_{(i,j) \in I \times J} X_i \times Y_j.$$

Dually there is a coproduct on pro-rings, called the completed tensor product. As an example, we see that $\widehat{\mathbb{A}}^2$ is represented by

$$\mathbb{Z}[\![x]\!] \widehat{\otimes} \mathbb{Z}[\![y]\!] \cong \mathbb{Z}[\![x, y]\!].$$

Now that we have finite products we can define formal groups, formal rings, etc.

Example 2.2.22. The additive formal group $\widehat{\mathbb{G}}_a$ takes a ring R to $\mathrm{Nil}^+(R)$, the additive group of nilpotent elements of R. Clearly its underlying formal scheme is isomorphic to $\widehat{\mathbb{A}}^1$. Fixing an isomorphism, then the additive group structure is defined by the following maps:

$$
\begin{array}{rclcl}
\epsilon_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z} & x & \mapsto & 0 \\
\Delta_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x_1, x_2]\!] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 \\
\chi_+ : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x]\!] & x & \mapsto & -x.
\end{array}
$$

Example 2.2.23. The multiplicative formal group $\widehat{\mathbb{G}}_m$ takes a ring R to the multiplicative group $(1 - \mathrm{Nil}(R))^\times$. Identifying this set with $\mathrm{Nil}(R)$ we see that $\widehat{\mathbb{G}}_m$ is isomorphic to $\widehat{\mathbb{A}}^1$ as a formal scheme. Fixing an isomorphism, the group structure is defined by the following maps:

$$
\begin{array}{rclcl}
\epsilon_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z} & x & \mapsto & 0 \\
\Delta_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x_1, x_2]\!] & x & \mapsto & x_1 \otimes 1 + 1 \otimes x_2 - x_1 \otimes x_2 \\
\chi_\times : \mathbb{Z}[\![x]\!] & \to & \mathbb{Z}[\![x]\!] & x & \mapsto & -\sum_{i \geq 0} x^{i+1}.
\end{array}
$$

Remark 2.2.24. Hazewinkel refers to the above formal group law as $\widehat{\mathbb{G}}_m^-$ [Haz78]. The multiplicative formal group is usually defined to represent the multiplicative group $(1 + \mathrm{Nil}(R))^\times$. However, the formal group law in 2.2.23 naturally occurs as an E_∞ orientation on complex K-theory, while the standard example does not [And95].

2.2.4 Cartier duality

Since our interest is actually in the ring that represents a given scheme, it is desirable to have a theory of duality for schemes that corresponds to taking the linear dual of the representing ring. Of course, for such a duality to exist we are going to need that the dual of the representing ring is another commutative unital ring, which means that the original scheme needs to be a commutative group scheme. In order for a group scheme to be canonically isomorphic to its double dual we are going to need that the representing ring be a dualizable module. Since we also want our theory of duality to apply to formal schemes we are going to need some assumptions about the maps in the pro-systems that define the representing pro-rings. The classical theory of Cartier duality (see [Dem72]), once suitably extended as in [Str00b, Section 6.4], suits our purposes.

Cartier duality is the analogue of Pontryagin duality for group schemes. The classical theory of Cartier duality requires that we work over a field and as a result safely ignores some of the issues described above. When working over more general rings we will need some new algebraic restrictions, to obtain a well-behaved duality theory.

We start by considering a suitable category of objects dual to k-algebras. Identifying k-algebras with the category of commutative monoids in the category of k-modules using the tensor product structure, we see that the appropriate dual is the category of cocommutative comonoids in the category of k-modules or, equivalently, counital, cocommutative coalgebras.

Definition 2.2.25. *Suppose U is a k-coalgebra free on a basis I. Let $\mathcal{D}_I$ be the category whose objects are the k-subcoalgebras of U which are free modules on a finite subset of I and whose morphisms are inclusions. If there exists a basis I such that*

$$U \cong \operatorname*{colim}_{\mathcal{D}_I} U_i$$

then we say I is a good basis *for U. Those coalgebras which admit a good basis will be called* basic*.*

Notation 2.2.26. *We denote the full subcategory of basic coalgebras in the category of coalgebras by $\mathcal{BCoAlg}$.*

Definition 2.2.27. *To a basic coalgebra $U = \operatorname{colim} U_i$, we define the formal scheme $\operatorname{Sch} U = \operatorname{colim} \operatorname{Spec} U_i^*$, where U_i^* is the linear dual $\operatorname{Mod}_k(U_i, k)$.*

Here $\lim U_i^*$ inherits its multiplicative structure from the coalgebra structure on U_i (see [Str00b, 4.59]). It follows that Sch does indeed define a functor from basic coalgebras to formal schemes whose image $\mathcal{CFSch}$, we call *coalgebraic formal schemes.*

We construct an inverse functor $c : \mathcal{CFSch} \to \mathcal{BCoAlg}$ by setting

$$c(\operatorname{colim} \operatorname{Spec} U_i^*) = \operatorname{colim} U_i^{**} \cong \operatorname{colim} U_i.$$

If our coalgebra U has the additional multiplicative structure making it a commutative Hopf algebra then $\operatorname{Spec} U$ ($\operatorname{Sch} U$) is a commutative (formal) group scheme. Given a formal coalgebraic group scheme

$$\widehat{\mathbb{G}} \cong \operatorname{colim} \operatorname{Spec} U_i^*,$$

we define the Cartier dual to be

$$\begin{aligned} D\widehat{\mathbb{G}} &= \operatorname{Spec} c\widehat{\mathbb{G}} \\ &\cong \operatorname{Spec} \operatorname{colim} U_i \\ &\cong \operatorname{Spec} U. \end{aligned}$$

Restricting to those coalgebraic formal schemes that are actually informal schemes we can apply D again to get

$$\begin{aligned} DD\widehat{\mathbb{G}} &= D \operatorname{Spec} c\widehat{\mathbb{G}} \\ &= c \operatorname{Spec} c\widehat{\mathbb{G}} \\ &\cong c \operatorname{Spec} U \\ &\cong \operatorname{colim} \operatorname{Spec} U_i^* \\ &\cong \widehat{\mathbb{G}}. \end{aligned}$$

This gives us a well-behaved duality on group schemes that has the effect of taking the linear dual on the representing rings.

Example 2.2.28. The linear dual of the truncated polynomial algebra $\mathbb{Z}[x]/(x^n - 1)$ with x grouplike (i.e., $\Delta x = x \otimes x$) is the module

$$\bigoplus_{i=0}^{n-1} \mathbb{Z} e_i$$

with coproduct

$$\Delta e_k = \sum_{i=0}^{n-1} e_i \otimes e_{\sigma(i)}$$

where $\sigma(i) \equiv k - i \pmod{n}$ and $0 \leq \sigma(i) < n$. The algebra structure is determined by the relations $e_i e_j = \delta_{ij} e_i$. We can see $\operatorname{Spec} \mathbb{Z}[x]/(x^n - 1)$ is the group scheme whose R-points are the multiplicative group of nth roots of unity in R (which might be trivial for a given R), while its Cartier dual is the constant functor $R \to \mathbb{Z}/n$.

Remark 2.2.29. For a coalgebraic commutative formal group scheme $\mathbb{H}$ one can define (see [Str00a, 4.69]) the following *scheme* of maps in commutative formal groups

$$D\mathbb{H} = \mathcal{GrpSch}(\mathbb{H}, \mathbb{G}_m).$$

This definition of the functor D most closely resembles the classical definition of Cartier duality and Strickland has shown that these definitions coincide [Str00a, 6.15].

2.3 Three perspectives

2.3.1 Symmetric schemes

We want to study operations on sets of monic polynomials. Some of these operations are easier to describe under the assumption that the monic polynomials split. As an intermediate step, we examine monic polynomials with a specified splitting, then monic polynomials that have a splitting before proceeding to general monic polynomials.

We will identify a *split* monic polynomial over R

$$\begin{aligned} f(x) &= \sum_{i=0}^{n} b_{n-i}x^i \\ &= \prod_{i=0}^{n}(x - t_i) \end{aligned}$$

with its *unordered* set of roots $\{t_1, \ldots, t_n\}$.

Definition 2.3.1. *Let the n^{th} splitting functor* Split_n *denote the functor from rings to sets that takes a ring R to the set of split monic polynomials with coefficients in R or, alternatively, the corresponding sets of roots of those polynomials.*

Remark 2.3.2. Note that Split_n is not a scheme, affine or otherwise, although it is related to a stack.

Definition 2.3.3. *Let the nth representable splitting functor* rSplit_n *be the affine scheme with the following R-points*

$$\mathrm{rSplit}_n(R) = \left\{ f(x) = \prod_{i=1}^{n}(x - t_i),\ t_i \in R \right\} \cong R^n.$$

Clearly rSplit_n is isomorphic to affine n-space and we have a natural transformation

$$U : \mathrm{rSplit}_n \to \mathrm{Split}_n$$

where we forget the ordering of the roots.

Definition 2.3.4. *A natural transformation*

$$G : \mathrm{Split}_n \to \mathrm{Split}_m,$$

is algebraic *if there exists a natural transformation* $\widetilde{G} : \mathrm{rSplit}_n \to \mathrm{rSplit}_m$ *making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{rSplit}_n & \overset{\widetilde{G}}{\dashrightarrow} & \mathrm{rSplit}_m \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathrm{Split}_n & \xrightarrow{G} & \mathrm{Split}_n \end{array}$$

Typically, we construct G from $\widetilde{G}$ by checking that $U\widetilde{G}$ factors through U, in which case we abuse notation and write G instead of $\widetilde{G}$ for the algebraic map. For example, we have the map

$$F_k : \mathrm{rSplit}_n \to \mathrm{rSplit}_{nk}$$

sending $(t_1, \ldots, t_n)$ to $([k]t_1, \ldots, [k]t_n)$, where $[k]t_i$ indicates repeat the root t_i k-times. Passing from $([k]t_1, \ldots, [k]t_n)$ to $\{[k]t_1, \ldots, [k]t_n\}$ we see that reordering the t_i's does not change the target set, so the map F_k descends to give an algebraic map. We define this map in terms of its R-points:

$$F_k : \sum_{i=0}^{n} b_{n-i}x^i = \prod_{i=1}^{n}(x - t_i) \mapsto \sum_{i=0}^{nk} b'_{nk-i}x^i = \prod_{i=0}^{n}(x - t_i)^k. \tag{2.3.5}$$

The coefficient b_i of the polynomial $f(x)$ can be identified with the elementary symmetric polynomial

$$(-1)^{n-i}\sigma_i(t_1, \ldots, t_n)$$

in the roots $t_1, \ldots, t_n$. Here the elementary symmetric functions $\sigma_i(t_1, \ldots, t_n)$ are defined by the following generating function

$$\prod_{i=1}^{n}(x + t_i) = \sum_{i=0}^{n} \sigma_{n-i} x^i.$$

If $i \leq n$ then $\sigma_i(t_1, \ldots, t_n) = \sigma_i(t_1, \ldots, t_n, 0)$, giving us well-defined elementary symmetric functions σ_i on "enough" variables.

Theorem 2.3.6 (Newton [Hus94])**.** *There is an isomorphism of algebras* $R[\sigma_1, \ldots, \sigma_n] \cong R[t_1, \ldots, t_n]^{\Sigma_n}$.

Since the coefficients b'_i in Equation 2.3.5 are symmetric in $t_1, \ldots, t_n$ they can be expressed as polynomials in the elementary symmetric functions or, equivalently, as polynomials $p_i(b_1, \ldots, b_n)$ in the coefficients of f. We can now drop the requirement that the polynomial splits and just use the polynomials p_i to define operations on monic polynomials.

Definition 2.3.7. *Let the nth symmetric scheme* $\mathrm{symm}_n \cong \mathrm{Spec}(\mathbb{Z}[b_1, \ldots, b_n])$ *be the scheme satisfying*

$$\mathrm{symm}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i} x^i \mid b_i \in R, b_0 = 1 \right\}$$

and

We also have formal analogues of the above schemes.

Definition 2.3.8. *Let the* n^{th} *formal splitting functor* $\widehat{\mathrm{Split}}_n$ *be the formal scheme satisfying*

$$\widehat{\mathrm{Split}}_n(R) = \left\{ f(x) \in \widehat{\mathrm{symm}}_n(R) \mid f(x) = \prod_{i=1}^{n}(x - t_i),\ t_i \in \mathrm{Nil}(R) \right\}.$$

Let the nth formal symmetric scheme $\widehat{\mathrm{symm}}_n \cong \mathrm{Spf}(\mathbb{Z}[\![b_1, \ldots, b_n]\!])$ *be the formal scheme satisfying*

$$\widehat{\mathrm{symm}}_n(R) = \left\{ f(x) = \sum_{i=0}^{n} b_{n-i} x^i \mid b_0 = 1 \textit{ and } b_i \in \mathrm{Nil}(R) \textit{ for } i > 0 \right\}.$$

Theorem 2.3.9 (Algebraic Splitting Theorem)**.**

1. *An algebraic transformation of functors* $\mathrm{Split}_n \to \mathrm{Split}_k$ *determines a map of schemes* $\mathrm{symm}_n \to \mathrm{symm}_k$.

2. *An algebraic transformation of functors* $\mathrm{Split}_i \times \mathrm{Split}_j \to \mathrm{Split}_k$ *determines a map of schemes* $\mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_k$.

3. *Two maps* $f_1, f_2 : \mathrm{symm}_m \to \mathrm{symm}_k$ *are equal if and only if* $f_1 U_m = f_2 U_m$ *where* $U_m : \mathrm{rSplit}_m \to \mathrm{symm}_m$ *is the forgetful map.*

4. *The same results hold for the formal analogues of the above schemes.*

Proof. The argument that F_k induces a natural transformation $\mathrm{symm}_n \to \mathrm{symm}_{nk}$ given above goes through *mutatis mutandis* to prove parts 1 and 2. Namely, in each of these cases we see that the natural transformations on split monic polynomials define polynomial maps on the coefficients which allow us to define natural transformations on monic polynomials.

Part 3 follows from the fact that U_m corresponds to the following *injective* map on representing rings

$$\begin{aligned} \mathbb{Z}[b_1, \ldots, b_m] &\to \mathbb{Z}[t_1, \ldots, t_m] \\ b_i &\mapsto \sigma_i(t_1, \ldots, t_m). \end{aligned}$$

□

The last claim allows us to deduce relations between maps by checking them on the representable (formal) splitting functors.

This allows us to define a panoply of natural transformations. Unless we say otherwise, for each of the following definitions there is an analogous version for the corresponding formal scheme.

Definition 2.3.10. *Applying Theorem 2.3.9, let* $\oplus_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \to \mathrm{symm}_{i+j}$ *be the algebraic map corresponding to*

$$\begin{aligned} \oplus_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j &\longrightarrow \mathrm{Split}_{i+j} \\ S \times T &\mapsto S \coprod T. \end{aligned}$$

In terms of R-points we have another simple description

$$\begin{aligned} \oplus_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j &\rightarrow \mathrm{symm}_{i+j} \\ (f(x), g(x)) &\mapsto f(x)g(x). \end{aligned}$$

Construction 2.3.11. *Given an operation $\mu : \mathrm{Split}_1 \times \mathrm{Split}_1 \rightarrow \mathrm{Split}_1$ (or equivalently $\mathrm{symm}_1 \times \mathrm{symm}_1 \rightarrow \mathrm{symm}_1$) we define operations $\mu_{i,j} : \mathrm{symm}_i \times \mathrm{symm}_j \rightarrow \mathrm{symm}_{ij}$ determined by*

$$\begin{aligned} \mu_{i,j} : \mathrm{Split}_i \times \mathrm{Split}_j &\rightarrow \mathrm{Split}_{ij} \\ S \times T &\mapsto \coprod_{(s,t) \in S \times T} \mu(s,t). \end{aligned}$$

When $\mu(r,t) = rt$ we will denote the operation $\mu_{i,j}$ by $\otimes_{i,j}$.

Proposition 2.3.12. *The operations $\mu_{i,j}$ defined above Construction 2.3.11 distribute over $\oplus_{i,j}$.*

Proof. This follows immediately from the definition of $\mu_{i,j}$ and Theorem 2.3.9. □

Definition 2.3.13. *Let the map $i_0 : \mathrm{symm}_0 \longrightarrow \mathrm{symm}_1$, or equivalently ($i_0 : \mathrm{Split}_0 \rightarrow \mathrm{Split}_1$), satisfying $* \mapsto \{0\}$. Similarly we have a map $i_1 : \mathrm{symm}_0 \rightarrow \mathrm{symm}_1$ (but not a map $\widehat{\mathrm{symm}}_0 \rightarrow \widehat{\mathrm{symm}}_1$) defined by $* \mapsto \{1\}$.*

For each n we have an inclusion $\iota : \widehat{\mathrm{symm}}_n \rightarrow \widehat{\mathrm{symm}}_{n+1}$ defined as the composite

$$\widehat{\mathrm{symm}}_n \cong \widehat{\mathrm{symm}}_n \times \widehat{\mathrm{symm}}_0 \xrightarrow{id \times i_0} \widehat{\mathrm{symm}}_n \times \widehat{\mathrm{symm}}_1 \xrightarrow{\oplus_{n,1}} \widehat{\mathrm{symm}}_{n+1}.$$

This map takes the monic polynomial $f(x)$ to $x \cdot f(x)$. Now taking a colimit over these maps inverts x and since the colimit of formal schemes agrees with the colimit in $\mathcal{S}et^{\mathcal{R}ing}$ setting $z = x^{-1}$ we obtain:

Definition 2.3.14. *The formal scheme $\widehat{\mathrm{symm}}^0 = \operatorname{colim} \widehat{\mathrm{symm}}_n \cong \operatorname{Spf} \mathbb{Z}[\![b_1, b_2, \ldots]\!]$, satisfies*

$$\widehat{\mathrm{symm}}^0(R) = \left\{ f(z) = \sum_{i=0}^{n} b_i z^i \mid b_0 = 1,\ n \in \mathbb{N},\ b_i \in \mathrm{Nil}(R) \text{ for } i > 0 \right\}.$$

On R-points the inclusions

$$\widehat{\text{symm}}_n \to \widehat{\text{symm}}^0$$

take $f(x)$ to $x^{-n} f(x)$. Since filtered colimits commute with finite products the compatible system of maps

$$\begin{array}{ccc} \widehat{\text{symm}}_i \times \widehat{\text{symm}}_j & \xrightarrow{\oplus_{i,j}} & \widehat{\text{symm}}_{i+j} \\ \downarrow{\scriptstyle \iota\times\iota} & & \downarrow{\scriptstyle \iota\circ\iota} \\ \widehat{\text{symm}}_{i+1} \times \widehat{\text{symm}}_{j+1} & \xrightarrow{\oplus_{i+1,j+1}} & \widehat{\text{symm}}_{i+j+2} \end{array}$$

defines an operation

$$\oplus : \widehat{\text{symm}}^0 \times \widehat{\text{symm}}^0 \to \widehat{\text{symm}}^0$$

which we combine with

$$i_0 : \widehat{\text{symm}}_0 \to \widehat{\text{symm}}_1 \to \widehat{\text{symm}}^0$$

to make $\widehat{\text{symm}}^0$ into a formal monoid scheme. The product on $\widehat{\text{symm}}^0$ corresponds to multiplication of polynomials.

Moreover, polynomials whose constant coefficient is one and other coefficients are nilpotent form a group under multiplication. Indeed, the multiplicative inverse of

$$f(z) = 1 + \sum_{i=1}^{n} b_i z^i$$

is a power series where the coefficient of z^{n+k} lies in

$$(b_1, \ldots, b_n)^k.$$

Since $b_i \in \text{Nil}(R)$ this ideal is zero for large k and we see that $1/f$ is actually a polynomial of the correct form. It follows that $\widehat{\text{symm}}^0$ is a formal group scheme.

Definition 2.3.15. *The positive symmetric scheme is the scheme*

$$\widehat{\text{symm}}^+ = \coprod_{i\geq 0} \widehat{\text{symm}}_i = \text{colim} \coprod_{0\leq i\leq n} \widehat{\text{symm}}_i$$

equipped with the rig structure defined by the maps $\oplus_{i,j}$ and $\otimes_{i,j}$ with the additive identity *given by the inclusion* $\widehat{\mathrm{symm}}_0 \to \widehat{\mathrm{symm}}^+$ *and the multiplicative identify given by* i_0 *followed by the inclusion* $\widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}^+$.

Definition 2.3.16. *Assembling the maps*

$$\iota : \widehat{\mathrm{symm}}_i \to \widehat{\mathrm{symm}_{i+1}}$$

into a map $\mathrm{symm}^+ \to \widehat{\mathrm{symm}}^+$, *we set*

$$\widehat{\mathrm{symm}} \equiv \mathrm{colim}\left[\widehat{\mathrm{symm}}_+ \to \widehat{\mathrm{symm}}_+ \to \dots\right].$$

Note that if we restrict to the 0 component of $\widehat{\mathrm{symm}}_+$ and then take colimits we obtain the same system defining $\widehat{\mathrm{symm}}^0$. Hence the zeroth component of $\widehat{\mathrm{symm}}$ is $\widehat{\mathrm{symm}}^0$ and this component inherits a multiplication from $\widehat{\mathrm{symm}}$.

Remark 2.3.17. There is nothing special about this multiplication. We can take any group operation on $\widehat{\mathrm{symm}}_1$ with unit i_0 and extend it to define rig schemes, ring schemes and rng schemes.

A formal group structure

$$F : \widehat{\mathrm{symm}}_1 \times \widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}_1$$

determines a map

$$F^s : \widehat{\mathrm{symm}}^+ \times \widehat{\mathrm{symm}}^+ \to \widehat{\mathrm{symm}}^+$$

that makes $\widehat{\mathrm{symm}}^+$ into a formal rig scheme. Using Construction 2.3.11 we can define maps

$$F_{i,j} : \widehat{\mathrm{symm}}_i \times \widehat{\mathrm{symm}}_j \to \widehat{\mathrm{symm}}_{ij}.$$

By construction these maps fit together and distribute over addition. This multiplicative inherits its unital and associativity properties from F.

The colimit in 2.3.16 can be identified with Grothendieck's group completion construction which makes $\widehat{\mathrm{symm}}$ into a formal ring scheme. After restricting this multiplicative structure to the 0 component we obtain:

Proposition 2.3.18. *A formal group structure*

$$F : \widehat{\mathrm{symm}}_1 \times \widehat{\mathrm{symm}}_1 \to \widehat{\mathrm{symm}}_1$$

determines a map

$$F^s : \widehat{\mathrm{symm}}^0 \times \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}^0$$

that makes $\widehat{\mathrm{symm}}^0$ *into a formal rng scheme.*

Since the endomorphisms of a commutative group object always form a (generally non-commutative) ring, every group object has a $\mathbb{Z}$-module structure. Under this $\mathbb{Z}$-action on a group object G with multiplication μ, a positive integer n corresponds to the composite

$$[n] : G \xrightarrow{\Delta^{n-1}} G^n \xrightarrow{\mu^{n-1}} G.$$

Proposition 2.3.19. *Let n be a positive integer and G a connected (formal) commutative group scheme over a ring R containing $\mathbb{Z}[1/n]$. Then the $\mathbb{Z}$-module structure described above extends to a $\mathbb{Z}[1/n]$-module structure.*

Proof. It suffices to show that $[n]$ is an isomorphism. On the Hopf-algebra representing G we see that, modulo decomposables, $[n]$ takes any indecomposable to n times itself (connectivity of our Hopf algebra is key here). Since n is invertible over our base ring, this map is an isomorphism. □

If $n = kl$ then $[n]$ factors as $[k] \circ [l]$. In the case $G = \widehat{sym}^0$, we can still obtain a nontrivial factorization when n is a prime p: $[p] = V_p \circ F_p$. Where V_p and F_p are the following:

Definition 2.3.20. *Let the kth Frobenius operation $F_k : \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}_0$ be the unique endomorphism satisfying*

$$F_k : (1 - az) \mapsto (1 - az^k).$$

Remark 2.3.21. We can formally factorize any degree n polynomial in $\widehat{\mathrm{symm}}^0$ into a product of linear factors like the above. So this map is indeed determined by its behavior on a linear term. Equivalently, we could have constructed this map using Theorem 2.3.9.

Definition 2.3.22. *Let the* k*th Verschiebung operation* $V_k : \widehat{\mathrm{symm}}^0 \to \widehat{\mathrm{symm}}_0$ *be the unique endomorphism satisfying*

$$V_k : (1 - az) \mapsto (1 - a^k z).$$

2.3.2 Lambda schemes

In this section we will examine the scheme Λ and its dual. The scheme Λ plays an important role in the theory of λ-rings which encode common structures in representation theory and algebraic topology, see [AT69, Knu73]).

Definition 2.3.23. *The Lambda-scheme* Λ *is the ring scheme whose underlying additive group scheme is defined by*

$$\Lambda(R) = (1 + tR[\![z]\!])^{\times}.$$

The multiplicative structure is more complicated and we will explain it below. This scheme can be represented by the ring $\mathrm{Sym} = \mathbb{Z}[b_1, b_2, \dots]$*, since a homomorphism* $f : \mathrm{Sym} \to R$ *is determined by where the* b_i *are mapped to under* f*. These elements determine a power series*

$$\sum_{i \geq 0} f(b_i) z^i, \tag{2.3.24}$$

where we adopt the useful convention $b_0 = 1$ *and therefore* $f(b_0) = 1$*. Under this correspondence the additive group is described by*

$$\begin{array}{rclcl}
\epsilon_+ : \mathrm{Sym} & \to & \mathbb{Z} & b_n & \mapsto & 0 \\
\Delta_+ : \mathrm{Sym} & \to & \mathrm{Sym} \otimes \mathrm{Sym} & b_n & \mapsto & \sum_{i=0}^{n} b_i \otimes b_{n-i} \\
\chi_+ : \mathrm{Sym} & \to & \mathrm{Sym} & b_n & \mapsto & -\sum_{i=0}^{n-1} \chi_+(b_i) b_{n-i}
\end{array}$$

for all $n \geq 1$.

Since Λ takes values in formal rings we might expect it to be an inverse limit of regular schemes and in fact it is. Since $\mathrm{Sym} \cong \operatorname{colim} \mathrm{Sym}_n$ where $\mathrm{Sym}_n = \mathbb{Z}[b_1, \dots, b_n]$, we obtain

$$\mathrm{symm} = \operatorname{Spec} \mathrm{Sym} = \mathcal{R}\!\mathit{ing}(\operatorname{colim} \mathrm{Sym}_n, -) \cong \lim \mathcal{R}\!\mathit{ing}(\mathrm{Sym}_n, -) = \lim \operatorname{Spec} \mathrm{Sym}_n.$$

While it is clear that the Λ is an informal analogue of $\widehat{\mathrm{symm}}^0$, we have the following result whose proof we defer until Section 2.4.

Theorem 2.3.25. *The Cartier dual of* $\widehat{\mathrm{symm}}^0$ *is* Λ.

The Frobenius and Verschiebung maps defined above on $\widehat{\mathrm{symm}}^0$ induce maps on the Cartier dual. The conventional names for the duals of F_k and V_k are V_k and F_k respectively, i.e., the Frobenius and Verschiebung are interchanged under Cartier duality. The reader can check that this definition agrees with the obvious informal analogues of Definitions 2.3.20 and 2.3.22.

2.3.3 Witt schemes

Witt schemes appear in many areas of mathematics, from starring roles in the classification of commutative group schemes and p-divisible groups ([Dem72]), to class field theory for fields of characteristic p (Witt's original purpose) and to cameo appearances in the Teichmuller embedding of finite fields into rings of characteristic zero. The role that Witt schemes (or more precisely, the truncated Witt schemes), play in commutative group schemes is reflected in the classification of bicommutative Hopf algebras and their characterization by Dieudonne modules ([Goe98, Sch70]) . Their role in constructing characteristic 0 lifts of finite fields leads to their appearance in the construction of Landweber exact formal group laws and their associated cohomology theories ([Rez98]).

The Witt scheme is a ring scheme whose underlying scheme is isomorphic to $\mathbb{A}^\infty$, just like Λ. In fact, there is an isomorphism of ring schemes between them. We will exploit this fact to circumvent defining the Witt scheme's ring structure directly and save us the trouble of restating a number of integrality lemmas (see [Haz78]). On representing rings, this isomorphism reflects a different choice of generators which are more convenient for some purposes. For example, the formulas for the primitive elements are simpler and satisfy some useful congruences. In Section 2.5 we will give formulas relating the choices of generators.

Definition 2.3.26. *The Witt scheme* $\mathbb{W}$ *has the underlying scheme* $\mathrm{Spec}(\mathbb{Z}[\theta_1, \theta_2 \dots])$, *and a ring scheme structure which will be given in Corollary 2.3.28. We identify an element*

$f \in \mathbb{W}(R)$ *with the power series*

$$\prod(1 - f(\theta_n)t^n)^{-1} = 1 + p_1(f)t + p_2(f)t^2 + \cdots \in 1 + tR[\![t]\!].$$

For example,

$$\begin{aligned} p_1(f) &= f(\theta_1) \\ p_2(f) &= f(\theta_1)^2 + f(\theta_2) \\ p_3(f) &= f(\theta_1)^3 + f(\theta_1)f(\theta_2) + f(\theta_3). \end{aligned}$$

Examining these formulas for the coefficients and comparing them to Equation 2.3.24 defining the R-points of Λ, we can find a formula for $f(\theta_i)$ in terms of the $f(b_i)$, and conversely, inductively. This leads us to the following theorem :

Theorem 2.3.27 (cf. [Haz78])**.** *There is an isomorphism of schemes* $\mathbb{W} \cong \Lambda$.

Proof. By the Yoneda lemma, the maps from $\mathbb{W}$ to Λ are in bijection with

$$\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots]) \cong \mathcal{R}ing(\mathbb{Z}[b_1, b_2, \dots], \mathbb{Z}[\theta_1, \theta_2, \dots]).$$

The power series

$$\prod(1 - \theta_n t^n)^{-1}$$

defines an element of $\Lambda(\mathbb{Z}[\theta_1, \theta_2, \dots])$ and hence a map f. This map gives rise to maps

$$f_n : \mathbb{Z}[b_1, \dots, b_n] \to \mathbb{Z}[\theta_1, \dots, \theta_n].$$

Each of these algebras admits an augmentation that sends each of the polynomial generators to 0. The induced map on indecomposables

$$f_n : (b_1, \dots, b_n)/(b_1, \dots, b_n)^2 \to (\theta_1, \dots, \theta_n)/(\theta_1, \dots, \theta_n)^2$$

is an isomorphism because we have the following isomorphism modulo decomposables

$$\begin{aligned}\prod_{1\le i\le n}(1-\theta_i t^i)^{-1} &\equiv \prod_{1\le i\le n}(1+\theta_i t^i)\\ &\equiv 1+\sum_{1\le i\le n}\theta_i t^i.\end{aligned}$$

Using powers of the augmentation ideals to define filtrations on these algebras, we have an induced isomorphism on the associated graded algebras

$$gr(\mathbb{Z}[b_1,\ldots,b_n]) \to gr(\mathbb{Z}[\theta_1,\ldots,\theta_n]).$$

With such a filtration the associated graded of a polynomial algebra is itself, so we have an isomorphism between the truncated algebras $\mathbb{Z}[b_1,\ldots,b_n]$ and $\mathbb{Z}[\theta_1,\ldots,\theta_n]$. We obtain the desired isomorphism by taking colimits. □

Corollary 2.3.28. *The Witt scheme* $\mathbb{W}$ *admits the structure of a ring-scheme such that the map in Theorem 2.3.27 is an isomorphism of ring-schemes.*

Remark 2.3.29. We can consider this to be a definition of the ring-scheme structure on $\mathbb{W}$.

Corollary 2.3.30. *There is an isomorphism of formal group schemes* $\widehat{\mathbb{W}} \cong \widehat{\mathrm{symm}}^0$.

2.4 Formal schemes arising from the cohomology of a space

Now we will try to apply the above theory to the co/homology of a space. We are particularly interested in those spaces and cohomology theories that are connected to formal groups (see [Ada95, Hop99, Rav00]).

Notation 2.4.1. *If E is a cohomology theory and X a space then $E^*(X)$ will always refer to the unreduced E-cohomology of X. The reduced cohomology theory will be denoted $\tilde{E}^*(X)$.*

Notation 2.4.2. *For a multiplicative cohomology theory E, we will adopt the standard convention of writing E_* for $E^{-*}(*)$.*

Definition 2.4.3. *A cohomology theory E is called even-periodic if*

1. *E is multiplicative (i.e., E takes values in graded rings).*

2. *$E_{\text{odd}} = 0$*

3. *There exists a unit $x \in E_2$.*

The standard examples include even-periodic ordinary cohomology HPR, complex K-theory K, even-periodic Morava K-theory $\overline{K(n)}$, the Morava E-theories E_n, and even-periodic complex cobordism MP. We typically recognize these theories by their coefficient rings:

$$\begin{aligned} HPR_* &\cong R[v, v^{-1}] \\ K_* &\cong \mathbb{Z}[v, v^{-1}] \\ \overline{K(n)}_* &\cong \mathbb{F}_{p^n}[v, v^{-1}] \\ E_{n*} &\cong \mathbb{W}_p(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][v, v^{-1}] \text{ (see Section 2.3.3)} \\ MP_* &\cong \mathbb{Z}[b_1, b_2, \ldots][v, v^{-1}], \end{aligned}$$

where the grading is determined by putting all of the generators in degree 0 except for v which lies in degree 2. A nice description of the properties of these cohomology theories can be found in [Hop99, Rez98].

For the remainder of this chapter E will always denote some even-periodic cohomology theory.

Recall that the tensor product operation on vector bundles restricts to give a group operation on isomorphism classes of line bundles, the unit coming from the one dimensional trivial bundle [1] and the inverse of a bundle η is given by the dual bundle η^*. Since $\mathbb{C}P^\infty$ is a model for $BU(1)$ the classifying space of 1-dimensional complex line-bundles we obtain a multiplication map

$$\mu_\otimes : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

that makes $\mathbb{C}P^\infty$ into a group object in $\mathcal{hTop}$, the derived category of topological spaces.

Standard calculations (see [Hop99]) show that for an even-periodic cohomology theory E, we have

$$E^0(\mathbb{C}P^\infty) \cong E_0[\![x]\!] \tag{2.4.4}$$

with the choice of isomorphism dependent on the choice of unit in Definition 2.4.3. We also have Kunneth isomorphisms

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E^0(\mathbb{C}P^\infty) \otimes_{E_0} \cdots \otimes_{E_0} E^0(\mathbb{C}P^\infty).$$

Fixing an isomorphism as in Equation 2.4.4 canonically determines an isomorphism

$$E^0((\mathbb{C}P^\infty)^{\times n}) \cong E_0[\![t_1, \ldots, t_n]\!].$$

The map $\mu_\otimes$ gives rise to the coproduct

$$\begin{array}{rcl} \Delta_\otimes : E^0(\mathbb{C}P^\infty) & \longrightarrow & E^0(\mathbb{C}P^\infty) \otimes_{E_0} E^0(\mathbb{C}P^\infty) \\ E^0[\![x]\!] & \longrightarrow & E^0[\![x, y]\!] \\ x & \mapsto & F_E(x, y). \end{array}$$

The formal power series $F_E(x, y)$ is the formal group law associated to E with a specified orientation (which determines the isomorphism in Equation 2.4.4). A different choice of isomorphism will give rise to a formal group law of the form $F_E(\lambda x, \lambda y)$, for some unit $\lambda \in E^0$.

Although the power series

$$\Delta_\otimes(x) = F_E(x, y)$$

is called a formal group law, the map $\Delta_\otimes$ actually defines a cogroup object in pro-rings. Passing to the opposite category of formal schemes allows us to reverse the arrows and recover a group object.

Definition 2.4.5. *Given a CW-complex X and an even-periodic cohomology theory E, we define the formal scheme X_E associated to X and E by*

$$X_E = \operatorname{colim} \operatorname{Spec} E^0(X_\alpha),$$

where the filtered system defining the colimit is given by the filtration of X by its finite subcomplexes X_α.

Remark 2.4.6. Note that X_E is a covariant functor of X, making the notation convenient for studying diagrams of spaces. Also note that X_E must not be confused with the Bousfield localization $X_{\langle E\rangle}$, of X with respect to a homology theory E.

If X is a finite-dimensional CW-complex then X_E is defined by a directed system with terminal object $\operatorname{Spec} E^0(X)$. It follows that X_E is isomorphic to the ordinary scheme $\operatorname{Spec} E^0(X)$.

Definition 2.4.7. *The formal group associated to an even-periodic cohomology theory E, $\widehat{\mathbb{G}}_E$ is the formal scheme $\mathbb{C}P^\infty_E$ over E_0, with the group structure induced by the tensor product of vector bundles.*

By well known calculations we can identify the formal group associated to K-theory with $\widehat{\mathbb{G}}_m$ from Example 2.2.9 and the formal group associated to ordinary cohomology $HP\mathbb{Z}$ with $\widehat{\mathbb{G}}_a$ from Example 2.2.8.

If X is a commutative H-group with $H_*(X)$ even and torsion-free then the relevant Atiyah-Hirzebruch and Kunneth spectral sequences collapse to show

$$\begin{aligned} E_*(X) &\cong E_* \otimes H_*(X) \\ E_*(X \times X) &\cong E_*(X) \otimes_{E_*} E_*(X) \end{aligned}$$

From this we can see that the H-group structure on X makes $\operatorname{Spec} E_* X$ a group scheme.

Since $H_* X$ is torsion-free and of finite type then similar arguments show

$$E^*(X) \cong \operatorname{Mod}_{E_*}(E_* X, E_*).$$

It now follows that:

Proposition 2.4.8. *Suppose X is a commutative H-group with $H_*(X)$ even, torsion free, and of finite type. Then $D \operatorname{Spec} E_0 X \cong X_E$.*

Now we recall several well known calculations (see, for example, [Swi02]).

Proposition 2.4.9. *The inclusion of a maximal torus*

$$(S^1)^{\times n} \to U(n)$$

induces a map

$$E^*(BU(n)) \to E^*((BS^1)^{\times n}) \cong E^*((\mathbb{C}P^\infty)^{\times n}) \cong E^*[\![t_1, \ldots, t_n]\!]$$

which lifts to an isomorphism

$$E^*(BU(n)) \cong E^*((\mathbb{C}P^\infty)^{\times n})^{\Sigma_n}.$$

Example 2.4.10. Combining this with Theorem 2.3.6 we see that $E^*BU(n) \cong E^*[\![\sigma_1, \ldots, \sigma_n]\!]$. It follows that we can identify $BU(n)_E$ with $\widehat{\mathrm{symm}}_n \times \operatorname{Spec} E^0$.

Proposition 2.4.11 ([AGP02, May99])**.** *The functor $K^0(-)$ is represented by the space $BU \times \mathbb{Z}$.*

The maps from $BU(i)$ into $BU(i+1)$, for each i, that classify adjoining a one dimensional trivial bundle to the universal i-dimensional bundle, induce a map

$$\coprod_{i \geq 0} BU(i) \to \coprod_{i \geq 0} BU(i)$$

that can be used to construct the following homotopy equivalence:

$$BU \times \mathbb{Z} \simeq \operatorname{hocolim} \left[\coprod_{i \geq 0} BU(i) \to \coprod_{i \geq 0} BU(i) \to \coprod_{i \geq 0} BU(i) \to \ldots \right].$$

Example 2.4.12. Combining Proposition 2.4.11 with 2.4.10 we see that we can identify $(BU \times \mathbb{Z})_E$ with $\widehat{\mathrm{symm}} \times \operatorname{Spec} E^0$ and $BU_E \equiv (BU \times \{0\})_E$ with $\widehat{\mathrm{symm}}^0 \times \operatorname{Spec} E^0$.

Using the construction of $\widehat{\mathrm{symm}}^0$ as the colimit of the $\widehat{\mathrm{symm}}_n$, or from the construction of BU given above, we see that

$$E^*BU \cong E^*[\![\sigma_1, \sigma_2, \ldots]\!].$$

Since BU satisfies the conditions of Proposition 2.4.8, we see that the Cartier dual of $\widehat{\mathrm{symm}}^0$ is represented by $\mathrm{Spec}(E_0BU)$. It is well known that, up to completion, the homology of BU is self-dual as a Hopf algebra. It follows that $\mathrm{Spec}(E_0BU)$ is the informal analogue of $\widehat{\mathrm{symm}}^0$ and

$$\mathrm{Spec}(E_0BU) \cong \Lambda \times \mathrm{Spec}\, E^0.$$

We can now consolidate the work above.

Theorem 2.4.1. *If E is an even-periodic ring spectrum then we have the following chain of group scheme isomorphisms:*

$$\begin{aligned} \mathrm{Spec}\, E_0BU \cong& \Lambda \times \mathrm{Spec}\, E^0 \\ \cong& \mathbb{W} \times \mathrm{Spec}\, E^0. \end{aligned}$$

We also have the following chain of formal group scheme isomorphisms:

$$\begin{aligned} BU_E \cong& \widehat{\mathrm{symm}}^0 \times \mathrm{Spec}\, E^0 \\ \cong& \widehat{\mathbb{W}} \times \mathrm{Spec}\, E^0 \equiv D\mathbb{W} \times \mathrm{Spec}\, E^0 \\ \cong& \widehat{\Lambda} \times \mathrm{Spec}\, E^0 \equiv D\Lambda \times \mathrm{Spec}\, E^0. \end{aligned}$$

Remark 2.4.13. We could have extended this chain of isomorphisms to connnect to the Burnside ring of $\widehat{\mathbb{Z}}$, the necklace algebra, or to the curves functor [Haz08] if it were not for constraints on time, space, and energy. These correspondences are too beautiful and common to avoid being rediscovered again and again; this is especially true for the author. The extensive bibliography (471 entries!) in [Haz08] is a testament to this.

We conclude this section with a simple application of the above theorem. When working of p-local ring the Cartier dual of the Witt scheme admits the following idempotent map of group schemes:

$$\epsilon = \sum_{\gcd(n,p)=1}^{\widehat{\mathbb{W}}} \left[\frac{\mu(n)}{n}\right] V_n F_n \tag{2.4.14}$$

where $\mu(n)$ is the Möbius function defined by the recurrence relation

$$\sum_{d|n} \mu(d) = \delta_{1,n}.$$

This idempotent map splits $\widehat{\mathbb{W}}$ into a countable product of group schemes. The image of the idempotent is denoted $\widehat{\mathbb{W}}_p$ which is Cartier dual to the p-Witt scheme.

This determines a splitting of each of the formal group schemes in Theorem 2.4.1. In particular, it determines a Hopf algebra splitting of E^*BU that corresponds to the generalized cohomology form of Husemoller's splitting [Hus71]. Applying Cartier duality we obtain a splitting of E_*BU and the image of the idempotent is self-dual (up to completion).

The essential part of the construction of this splitting, is the identification of E_*BU with the free commutative algebra $\tilde{E}_*(\mathbb{C}P^\infty)$. This is dual to the statement that E^*BU is the cofree coalgebra on $\tilde{E}^*(\mathbb{C}P^\infty)$, which is a reformulation of the splitting principle for E^*BU.

One can also see that Equation 2.4.14 is exactly the formula for Quillen's idempotent [Qui69a, 7] on curves in a formal group, which he used to split $MU_{(p)}$. Indeed both splittings are constructed in the same way.

2.5 The Chern classes of a tensor product of arbitrary vector bundles

Suppose we have two 3-dimensional complex vector bundles over some fixed space which have E-theory Chern classes a_1, a_2, a_3 and b_1, b_2, b_3 respectively. The tensor product of these two bundles is a 9-dimensional vector bundle and this operation defines a map of algebras:

$$\Delta : E^*(BU(9)) \cong E^*[\![c_1, \ldots, c_9]\!] \to E^*[\![a_1, a_2, a_3]\!] \otimes_{E^*} E^*[\![b_1, b_2, b_3]\!].$$

By our form of the splitting principle (Theorem 2.3.9), we obtain formulas for this map (Δ corresponds to $\otimes_{3,3}$). Even for such a small example the formulas already are quite complicated. Computing the coproduct of higher Chern classes is greatly facilitated by using a computer and we have implemented our calculations in Mathematica, although it is straightforward to implement them in any symbolic computer package. In the first two examples below, we have grouped the terms together to emphasize the symmetry in the expansions.

Due to obvious limitations on space and the reader's assumed interest, we have only included the first few coproducts in each of the cases below. The author is not aware of such formulas ever appearing in print and we record them for posterity, although our primary interest is in demonstrating that through a limited range these formulas are computable by the above methods.

The simplest possible example is when E is $H\mathbb{Z}$. In this case we are working with the additive formal group law described above. In this particular case, one can find these formulas (in an unexpanded form) in [MS74, p. 87-88].

$$\begin{aligned}
\Delta c_1 =&\ 3\,(a_1 \otimes 1 + 1 \otimes b_1)\\
\Delta c_2 =&\ 3\left((a_2 + a_1^2) \otimes 1 + 1 \otimes (b_2 + b_1^2)\right) + 8a_1 \otimes b_1\\
\Delta c_3 =&\ (a_1^3 + 6a_1a_2 + 3a_3) \otimes 1 + 1 \otimes (b_1^3 + 6b_1b_2 + 3b_3)\\
&+ 7\left(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\right)\\
\Delta c_4 =&\ + 3\left((a_1^2a_2 + a_2^2 + 2a_1a_3) \otimes 1 + 1 \otimes (b_2^2 + b_1^2b_2 + 2b_1b_3)\right)\\
&+ 2\left((6a_1a_2 + 3a_3 + a_1^3) \otimes b_1 + a_1 \otimes (6b_1b_2 + 3b_3 + b_1^3)\right)\\
&+ 3a_2 \otimes b_2 + 5a_1^2 \otimes b_1^2\\
&+ 6\left(a_1^2 \otimes b_2) + a_2 \otimes b_1^2\right)
\end{aligned}$$

When $E = KU$ is equipped with the (E_∞) orientation described above we obtain the following coproduct formulas (here $u \in K^{-2}$ is the Bott element):

$$\begin{aligned}
\Delta c_1 =& 3(a_1 \otimes 1 + 1 \otimes b_1) - u a_1 \otimes b_1 \\
\Delta c_2 =& 3\Big((a_1^2 + a_2) \otimes 1 + 1 \otimes (b_1^2 + b_2)\Big) + 8 a_1 \otimes b_1 \\
& - 2u\Big(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\Big) \\
& + u^2\Big(a_1^2 \otimes b_2 - 2a_2 \otimes b_2 + a_2 \otimes b_1^2\Big) \\
\Delta c_3 =& a_1^3 \otimes 1 + 1 \otimes b_1^3 + 6(a_1 a_2 \otimes 1 + 1 \otimes b_1 b_2) + 3(a_3 \otimes 1 + 1 \otimes b_3) \\
& + 7\Big(a_1 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1\Big) \\
& + u\Big[-(a_1^3 + 6a_1 a_2 + 3a_3) \otimes b_1 - a_1 \otimes (b_1^3 + 6b_1 b_2 + 3b_3) \\
& \quad -2(a_1^2 \otimes (b_1^2 + b_2) + (a_1^2 + a_2) \otimes b_1^2) - 8a_2 \otimes b_2\Big] \\
& + u^2\Big[a_1^3 \otimes b_2 + a_2 \otimes b_1^3 + 2(a_1 a_2 \otimes b_1^2 + a_1^2 \otimes b_1 b_2) \\
& \quad +3(a_3 \otimes b_1^2 - 2a_2 \otimes b_3 - 2a_3 \otimes b_2 + a_1^2 \otimes b_3)\Big] \\
& + u^3\Big[-(a_1^3 \otimes b_3 + a_1 a_2 \otimes b_1 b_2 + a_3 \otimes b_1^3) \\
& \quad +3(a_1 a_2 \otimes b_3 - a_3 \otimes b_3 + a_3 \otimes b_1 b_2)]
\end{aligned}$$

The above two complex oriented theories have the only formal group laws (up to isomorphism) that are finite. When working with another theory we are forced to use finite expansions.

The formulas below are based on an expansion of the formal group law for BP, at the prime 3, out to the 4th power in the first Chern class. We have chosen to use the Hazewinkel generators since they provide a formal group law with *integral* coefficients. Using these choices we obtain the following coproduct formulas for the tensor product of two 2-dimensional vector bundles.:

$$
\begin{aligned}
\Delta c_1 =& 2\left(a_1 \otimes 1 + v_1(a_1 \otimes b_2 + a_2 \otimes b_1) + 1 \otimes b_1\right) \\
& - v_1\left(a_1 \otimes b_1^2 + a_1^2 \otimes b_1\right) \\
\Delta c_2 =& -5v_1^2 a_1 a_2 \otimes b_1 b_2 - v_1 a_1 \otimes b_1^3 + v_1 a_1 a_2 \otimes b_1 + v_1 a_1 \otimes b_1 b_2 \\
& + v_1^2 a_1 a_2 \otimes b_1^3 - 2v_1 a_1^2 \otimes b_1^2 + v_1^2 a_1^2 \otimes b_2^2 - v_1 a_1^3 \otimes b_1 \\
& + v_1^2 a_1^3 \otimes b_1 b_2 + v_1^2 a_1^4 \otimes b_2 - 4v_1^2 a_2 a_1^2 \otimes \ b_2 + 8v_1 a_2 \otimes b_2 \\
& - 4v_1^2 a_2 \otimes b_1^2 b_2 + 2v_1^2 a_2 \otimes b_2^2 + v_1^2 a_2 \otimes b_1^4 + 2v_1^2 a_2^2 \otimes b_2 \\
& + v_1^2 a_2^2 \otimes b_1^2 + 3a_1 \otimes b_1 + a_1^2 \otimes 1 + 2a_2 \otimes 1 + 1 \otimes b_1^2 + 2 \otimes b_2 \\
\Delta c_3 =& -v_1 a_1 a_2 \otimes b_1^2 + 2v_1 a_1 a_2 b_2 + 2v_1^2 a_1 a_2 \otimes b_2^2 \\
& - v_1^3 a_1 a_2 \otimes b_1^2 b_2^2 + 2v_1^3 a_1 a_2 \otimes b_2^3 - 4v_1^2 a_1 a_2 \otimes b_1^2 b_2 \\
& - 6v_1^2 a_1 a_2^2 \otimes b_2 + 2v_1^3 a_1 a_2^2 \otimes b_1^2 b_2 + a_1 \otimes b_1^2 + 2a_1 \otimes b_2 \\
& - 2v_1 a_1 \otimes b_1^2 b_2 + 2v_1 a_1 \otimes b_2^2 + v_1^2 a_1 a_2 \otimes b_1^4 - v_1^3 a_1^2 a_2^2 \otimes b_1 b_2 \\
& + a_1^2 \otimes b_1 - v_1 a_1^2 \otimes b_1^3 - v_1 a_1^2 b_1 b_2 + v_1^2 a_1^2 a_2 \otimes b_1^3 + v_1^2 a_1^2 \otimes b_1 b_2^2 \\
& + 2v_1^3 a_1^3 a_2 \otimes b_2^2 - v_1 a_1^3 \otimes b_1^2 + v_1^2 a_1^3 \otimes b_1^2 b_2 + v_1^2 a_1^4 \otimes b_1 b_2 \\
& - 2v_1 a_2 \otimes a_1^2 \otimes b_1 - 4v_1^2 a_2 a_1^2 \otimes b_1 b_2 - v_1^3 a_1^2 a_2 \otimes b_1^3 b_2 \\
& + 2v_1^3 a_1^2 a_2 \otimes b_1 b_2^2 + 2v_1^2 a_1^3 a_2 \otimes b_2 - v_1^3 a_1^3 a_2 \otimes b_1^2 b_2 \\
& + 2a_2 \otimes b_1 + 2v_1 a_2 \otimes b_1 b_2 + 2v_1^2 a_2 \otimes b_1^3 b_2 \\
& - 6v_1^2 a_2 \otimes b_1 b_2^2 - 6v_1^3 a_1 a_2^2 \otimes b_2^2 + 2v_1 a_2^2 \otimes b_1 + 2v_1^2 a_2^2 \otimes b_1 b_2 \\
& + 2v_1^3 a_2^2 \otimes b_1^3 b_2 - 6v_1^3 a_2^2 \otimes b_1 b_2^2 + v_1^2 a_1 a_2^2 \otimes b_1^2 \\
& + 2v_1^3 a_2^3 \otimes b_1 b_2 + 2a_1 a_2 \otimes 1 + 2 \otimes b_1 b_2
\end{aligned}
$$

CHAPTER 3
H_∞ ORIENTATIONS ON BP

3.1 Introduction

"I have transformed the problem from an intractably difficult and possibly quite insoluble conundrum into a mere linguistic puzzle. Albeit," he muttered, after a long moment of silent pondering, "an intractably difficult and possibly insoluble one."

- D. Adams

This paper arose out of the authors' attempts to answer the long-standing open question:

Does the Brown-Peterson spectrum admit an E_∞ or H_∞ ring structure?

We offer a partial answer to an intimately related question:

Does the Brown-Peterson spectrum admit an H_∞ ring structure compatible with the H_∞ ring structure on MU?

At the primes 2 and 3, the answer is no. At these primes, this implies that BP does not admit the structure of a commutative MU-algebra.

When making computations in homotopy theory, it is convenient to work p-locally. Quillen constructed a p-local, idempotent map

$$\varepsilon : MU_{(p)} \to MU_{(p)}$$

that splits $MU_{(p)}$ into a wedge of suspensions of the Brown-Peterson spectrum BP. More specifically, we have the following:

Theorem 3.1.1 ([Qui69b]). *There is a map $r : MU_{(p)} \to BP$ classifying a universal p-typical formal group law. This map admits a section s, and the composite sr is the map ε above.*

As a consequence, for many calculations there is no loss of information when replacing $MU_{(p)}$ with BP. This is often desirable given that the Brown-Peterson spectrum has a much smaller coefficient ring and is more amenable to computations. For these reasons one usually makes computations using BP rather than $MU_{(p)}$.

The category of E_∞ ring spectra is equivalent to the subcategory of D-algebras in the category of spectra for a suitable monad D, while the category of H_∞ ring spectra is equivalent to the subcategory of D-algebras in the *homotopy* category of spectra.[1]

It follows that every E_∞ ring spectrum becomes an H_∞ ring spectrum after passing to the homotopy category. Elmendorf-Kriz-Mandell-May [EKMM97] show that E_∞ ring spectra are weakly equivalent to commutative S-algebras (*i.e.*, commutative monoids in the category of S-modules). They go on to show that many constructions in algebra can be mimicked in the categories of modules over an S-algebra. This correspondence requires a deep point-set level theory of spectra, but has dramatically improved our understanding of the stable homotopy category.

Although the notion of an H_∞ ring structure is too weak to perform such constructions, Bruner-May-McClure-Steinberger [BMMS86] have shown that a spectrum with an H_∞ ring structure is equivalent to a cohomology theory with a well-behaved theory of power operations in degree 0 and this is enough for many applications. They also develop the theory of H_∞^d ring spectra, which have a larger family of power operations.

For the spectrum MU, the underlying geometry of cobordism endows it with a standard E_∞ ring structure [May77]. The associated H_∞ ring structure comes from the H_∞^2 structure constructed by tom Dieck [tD68]. These structures descend to E_∞ and H_∞^2 structures on $MU_{(p)}$.

A map of H_∞ ring spectra is a natural transformation on cohomology functors that respects power operations. Since computations in BP-cohomology rely heavily on the retraction of $MU_{(p)}$ onto BP, a useful H_∞ ring structure should appear in this way.

Since the construction of BP involves maps into and out of MU, it is natural to ask if either map can be made into a map of H_∞ ring spectra. The first case has already been considered by Hu-Kriz-May:

1. This is not the homotopy category of D-algebras, whose underlying objects are E_∞ ring spectra.

Theorem 3.1.2 (See [HKM01, 2.11] and [BM04, App. B])**.** *There are no H_∞ ring maps from BP to $MU_{(p)}$.* [2]

Our work concerns maps out of MU into BP. Taking [BMMS86] as a starting point, we directly address the question of H_∞^2 ring structure for BP under MU and deduce the non-existence of H_∞ ring structure under MU as a consequence.

Our main result (Theorem 3.2.2) is that, at the primes 2 and 3, there are no H_∞ ring maps from MU to BP which endow BP with a universal p-typical formal group law. The precise statement and proof are given in Section 3.2.

In addition to some of our own reductions, this theorem relies on some results of McClure [BMMS86, VIII] and his formula for possible obstructions to H_∞^2 structure. We outline the reductions in Section 3.2 and give an alternate formula for McClure's obstruction in Section 3.5. Section 3.3 reviews E_∞, H_∞, and H_∞^2 ring spectra. In Section 3.4 we give an equivalent formulation of the problem in the language of formal group laws.

Using our formula for McClure's obstructions, we partially compute the obstructions at the primes 2 and 3 in Section 3.6 and show that Quillen's map is not a map of H_∞ ring spectra. Using this calculation we deduce that there are no maps of H_∞ ring spectra from $MU_{(p)}$ to BP when p equals 2 or 3.

Throughout this paper we will refer to a map of ring spectra $MU \to E$ as a (complex) orientation on E. For convenience, we will assume all spectra are localized at a prime p. We will also use the usual shorthand $E^* \equiv E^*(*)$ for the E cohomology of a point.

3.2 Main Theorems

Definition 3.2.1. *An orientation $f : MU \to E$ is called p-universal if for any orientation*

$$g : MU \to F$$

that defines a p-typical formal group law on F, there is a unique map $h : E \to F$ up to homotopy such that $g = hf$.

---

2. Although the cited theorem is stated in more restrictive terms, their proof directly applies to this more general result.

Definition 3.2.1 implies that all spectra admitting a p-universal orientation are canonically homotopy equivalent under MU. The map r in Theorem 3.1.1 is the standard example of a p-universal orientation on BP, so any E admitting a p-universal orientation can be thought of as an alternative form of the Brown-Peterson spectrum. The main result is as follows:

Theorem 3.2.2. *At the primes 2 and 3, there are no p-universal orientations of H_∞ ring spectra from MU to BP.*

Corollary 3.2.3. *At the primes 2 and 3, there are no p-universal orientations of BP that make it a commutative MU-algebra.*

We also have a reinterpretation of this result in the language of formal group laws.

Corollary 3.2.4 (Theorem 3.4.18)**.** *At the primes 2 and 3, the formal group law $\mathcal{VP}$ (Proposition 3.4.17) over $BP^{BC_p}[\chi^{-1}]$ is not p-typical.*

We do not see anything special about the primes 2 and 3 here, and we conjecture that the results above hold for all primes. We organize the proof of Theorem 3.2.2 into the following steps. The relevant definitions and notation are given in Sections 3.3 and 3.5.

Step 1 (Observation 3.3.8)**.** *If there is an H_∞ p-universal orientation on any spectrum E, then there is an H_∞ ring structure on BP such that Quillen's map r is a map of H_∞ ring spectra.*

Step 2 (Theorem 3.3.17)**.** *Quillen's orientation on BP is H_∞ if and only if it is H_∞^2; that is, if and only if for all n and each $\pi \leqslant \Sigma_n$, there are compatible power operations $\mathcal{P}_{\pi,BP}$ such that the diagram in Figure 3.1 commutes naturally in X.*

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{\mathcal{P}_{\pi,MU}} & MU^{2n*}(D_\pi X) \\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\ BP^{2*}(X) & \xrightarrow{\mathcal{P}_{\pi,BP}} & BP^{2n*}(D_\pi X) \end{array}$$

Figure 3.1: Naturality for an H_∞^2 Orientation on BP

For the sake of the reader, we outline the remaining sequence of reductions due to McClure [BMMS86, VIII]:

Step 3 ([BMMS86, VIII, p. 287])**.** *For all n and each $\pi \leqslant \Sigma_n$ there is at most one map $P_{\pi,BP}$ such that the diagram in Figure 3.1 commutes.*

Moreover, if all such diagrams commute then this system of maps defines an H^2_∞ ring structure on BP and r is a map of H^2_∞ ring spectra.

Step 4 ([BMMS86, VIII.7.2])**.** *The diagram in Figure 3.1 commutes naturally in X for all n and $\pi \leqslant \Sigma_n$ if and only if it commutes when $\pi = \Sigma_p$.*

Step 5 ([BMMS86, VIII.7.6])**.** *The diagram in Figure 3.1 commutes naturally in X for all n and $\pi \leqslant \Sigma_n$ if and only if it commutes when $\pi = \Sigma_p$ and $X = \mathbb{C}P^\infty$.*

Step 6 ([BMMS86, VIII.7.3])**.** *The diagram in Figure 3.1 commutes when $X = \mathbb{C}P^\infty$ and $\pi = \Sigma_p$ if and only if it commutes when the diagram in Figure 3.2 commutes.*

$$\begin{array}{ccc} MU^{2*}(\mathbb{C}P^\infty) & \xrightarrow{P_{C_p,MU}} & MU^{2p*}(BC_p \times \mathbb{C}P^\infty) \\ \downarrow r_* & & \downarrow r_* \\ BP^{2*}(\mathbb{C}P^\infty) & \xrightarrow{P_{C_p,BP}} & BP^{2p*}(BC_p \times \mathbb{C}P^\infty) \end{array}$$

Figure 3.2: Reduction to P_{C_p}

Notation 3.2.5. *For a formal group law F the p-series $[p]_F$ and the reduced p-series $\langle p \rangle_F$ are defined by the following equation.*

$$\overbrace{x +_F \cdots +_F x}^{p \text{ times}} = [p]_F\, x = x \cdot \langle p \rangle_F\, x$$

When clear from the context, we will drop the subscript F.

For a complex oriented cohomology theory E, let q_* denote the projection:

$$q_* : E^*BC_p \cong E^*[\![\xi]\!]/[p]\xi \to E^*[\![\xi]\!]/\langle p \rangle \xi. \tag{3.2.6}$$

Step 7 ([BMMS86, VIII.7.7])**.** *The diagram in Figure 3.2 commutes if and only if the diagram in Figure 3.3 commutes.*

$$\begin{array}{ccc} MU^{2*}(*) & \xrightarrow{qP_{C_p,MU}} & MU^{2p*}(BC_p)/\langle p\rangle\xi \\ {\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\ BP^{2*}(*) & \xrightarrow{qP_{C_p,BP}} & BP^{2p*}(BC_p)/\langle p\rangle\xi \end{array}$$

Figure 3.3: Reduction to a point

Since all of the rings in Figure 3.3 are torsion-free, it suffices to check commutativity of this diagram on the rational polynomial generators $[\mathbb{C}P^n] \in MU^{-2n}(*)$ (see Proposition 3.4.11.) After noting that a particular Euler class

$$\chi \in BP^{2(p-1)}[\![\xi]\!]/\langle p\rangle\xi$$

is not a zero divisor, we obtain our final reduction:

Step 8 ([BMMS86, VIII.7.8], Theorem 3.5.19)**.** *The diagram in Figure 3.3 commutes if and only if the elements*

$$MC_n(\xi) = r_* q_* \chi^{2n} P_{C_p,MU}[\mathbb{C}P^n] \in BP^{-2n(p-2)}[\![\xi]\!]/\left\langle p\right\rangle \xi$$

are 0 when $n \neq p^i - 1$ *for some* i.

In Theorem 3.4.18 we give an alternate statement of this result in the language of formal group laws.

A formula for the MC_n was first given by McClure in [BMMS86, VIII.7.8]. In Sections 3.5.1 and 3.5.3 we provide the relevant notation and summarize the derivation of this formula and then go on to prove the following formula:

$$MC_n(\xi) = \chi^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k].$$

Using this formula we give computations in Section 3.6 that yield the following result which, by the reductions above, proves Theorem 3.2.2.

Theorem 3.2.7.

1. *When* $p = 2$, $MC_2(\xi) \neq 0$.

2. *When* $p = 3$, $MC_4(\xi) \neq 0$.

3.3 E_∞ and H_∞ Ring Spectra

Let $\mathcal{S}$ denote the Lewis-May-Steinberger category of coordinate-free spectra and $h\mathcal{S}$ the stable homotopy category.

A spectrum in this category is indexed by finite dimensional subspaces of some countable inner product space $\mathcal{U}$. Let π be a subgroup of Σ_n the symmetric group on n letters. The space of linear isometries $\mathcal{L}(\mathcal{U}^n, \mathcal{U})$ is a free contractible Σ_n-space and by restriction a free contractible π-space which we will denote $E\pi$.

For each subgroup π of Σ_n we have an extended power functor on unbased spaces, based spaces, and spectra. For an unbased space Z, a based space W, and a spectrum X, the definitions are

$$D_\pi Z = E\pi \times_\pi Z^{\times n}$$
$$D_\pi W = E\pi_+ \wedge_\pi W^{\wedge n}$$
$$D_\pi X = E\pi \ltimes_\pi X^{\wedge n}.$$

where $\ltimes$ is the twisted half-smash product of [LMS86]. The functor from unbased to based spaces given by adjoining a disjoint basepoint relates the extended cartesian power on unbased spaces and the extended smash power on based spaces. For an unbased space Z, we have a homeomorphism of based spaces,

$$D_\pi(Z_+) \cong (D_\pi Z)_+.$$

In this paper we study power operations on unreduced cohomology theories, and therefore focus on unbased rather than based spaces. The extended cartesian power on unbased spaces is related to the extended smash power on spectra by the following: For an unbased space Z

$$D_\pi \Sigma^\infty_+(Z) = D_\pi \Sigma^\infty(Z_+) \cong \Sigma^\infty D_\pi(Z_+) \cong \Sigma^\infty (D_\pi Z)_+ = \Sigma^\infty_+ D_\pi Z. \tag{3.3.1}$$

With Equation 3.3.1 in mind, we may implicitly apply the functor Σ^∞_+ and will use the notation $D_\pi Z$ to denote either an unbased space or a spectrum, as determined by context.

The extended power construction, D_π is multiplicative, but not additive. For spectra X and Y, $D_{\Sigma_n}(X \vee Y)$ is given by the following formula [BMMS86, II.1.1]:

$$D_{\Sigma_n}(X \vee Y) = \bigvee_{i+j=n} D_{\Sigma_i} X \wedge D_{\Sigma_j} Y. \tag{3.3.2}$$

Definition 3.3.3. *Let D be the functor on $\mathcal{S}$ such that*

$$DX = \bigvee_{n \geq 0} D_{\Sigma_n} X.$$

The following result is standard (see [Rez98] for example).

Proposition 3.3.4. *There are natural transformations*

$$\mu : D^2 \to D$$
$$\eta : Id \to D$$

that make D a monad on $\mathcal{S}$.

Definition 3.3.5. *The category of E_∞ ring spectra is the category of D-algebras in $\mathcal{S}$.*

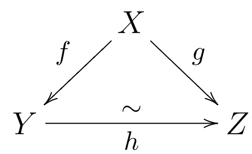
Proposition 3.3.6. *The monad D on $\mathcal{S}$ descends to a monad $\widetilde{D}$ on the stable homotopy category $h\mathcal{S}$.*

Proof. In [LMS86] it is shown that this functor preserves homotopy equivalences between cell spectra and takes cellular spectra to cellular spectra. It follows that D has a well-defined functor on the stable homotopy category, modeled by cellular spectra with homotopy classes of maps and that the previous structure maps pass to the stable category. □

Definition 3.3.7. *The category of H_∞ ring spectra is the category of $\widetilde{D}$-algebras in $h\mathcal{S}$.*

Step 1 now follows from:

Observation 3.3.8. *Consider the following diagram:*



If g is a map of $\widetilde{D}$-algebras for some monad $\widetilde{D}$ and h is an isomorphism, then h induces a $\widetilde{D}$ algebra structure on Y such that f and h are maps of $\widetilde{D}$-algebras.

Proposition 3.3.9. *Let $\Gamma : \mathcal{S} \to h\mathcal{S}$ denote the canonical functor. If X is an E_∞ ring spectrum, then ΓX is an H_∞ ring spectrum.*

Remark 3.3.10. Nearly all known H_∞ ring spectra are in the image of Γ. In Chapter 4the second author shows that the counterexample to the transfer conjecture constructed by Kraines-Lada [KL79] can be used to construct an example of a spectrum with an H_∞ ring structure which does not come from an E_∞ ring structure.

Definition 3.3.11. *Suppose E is an H_∞ ring spectrum, X is a spectrum, and*

$$f : X \to E$$

is a map representing a cohomology class in $E^0(X)$.

Define the π^{th} external cohomology operation

$$\mathcal{P}_{\pi,E} : E^0(X) \to E^0(D_\pi X)$$

by

$$(X \xrightarrow{f} E) \mapsto (D_\pi X \xrightarrow{D_\pi f} D_\pi E \to D_{\Sigma_n} E \hookrightarrow DE \xrightarrow{\mu} E).$$

If Y is a space, $Y^{\times n}$ is equipped with the π action induced by the inclusion $\pi \to \Sigma_n$. If we regard Y as a trivial π-space, then the diagonal map

$$\Delta : Y \to Y^{\times n}$$

is π-equivariant.

Definition 3.3.12. *Suppose E is an H_∞ ring spectrum, Y is a space. Define the map $\delta : B\pi \times Y \to D_\pi Y$ to be the following composite:*

$$\delta : (B\pi \times Y) \simeq E\pi \times_\pi Y \xrightarrow{E\pi \times \Delta} E\pi \times_\pi Y^n \cong D_\pi Y.$$

Define the π^{th} internal cohomology operation $P_{\pi,E} : E^0(Y) \to E^0(B\pi \times Y)$ as the composite

$$E^0(Y) \xrightarrow{\mathcal{P}_{\pi,E}} E^0(D_\pi Y) \xrightarrow{\delta^*} E^0(B\pi \times Y).$$

Notation 3.3.13. *We will drop the subscript E from the power operations $\mathcal{P}_{\pi,E}$ and $P_{\pi,E}$, when it is clear from the context.*

Definition 3.3.14 ([BMMS86, I.4.3])**.** *An H_∞^d ring structure on a spectrum E is a compatible family of maps*

$$D_{\Sigma_n}\Sigma^{di}E \to \Sigma^{din}E$$

for all $i \in \mathbb{Z}$.

When $i = 0$, these maps define an H_∞ structure on E. The compatibility conditions are graded analogs of those for an H_∞ ring spectrum, but are more easily understood in terms of the larger class of power operations on such a spectrum.

Let X be a spectrum and Y a space. For each $\pi \leqslant \Sigma_n$ and for each integer i, an H_∞^d spectrum E is equipped with the following power operations

$$\mathcal{P}_{\pi,E} : E^{di}(X) \to E^{din}(D_\pi X)$$
$$P_{\pi,E} : E^{di}(X) \to E^{din}(B\pi \times X).$$

The compatibility conditions for power operations on an H_∞^d ring spectrum are merely the appropriately graded analogues of those for an H_∞ ring spectrum. In particular, when $i = 0$, these are simply the power operations for the underlying H_∞ ring structure on E.

Remark 3.3.15. Note that the alternate definition of H_∞^d structure given in [BMMS86, II.1.3] is flawed. The argument that an H_∞^d ring structure on E determines an H_∞ ring structure on

$$\bigvee_{i \in \mathbb{Z}} \Sigma^{di} E$$

is correct, but the given argument for the converse is not. We suspect the converse is false.

Maps of H_∞^d ring spectra are those which commute with the structure maps in Definition 3.3.14. It follows that the category of H_∞^d ring spectra is a subcategory of the category of H_∞ ring spectra.

3.3.1 The Thom isomorphism and H_∞^2 orientations

Let V_k denote the standard representation of Σ_k on $\mathbb{C}^k$ and recall (*e.g.* [LMS86, Ch. X]) that

$$D_{\Sigma_k} S^{2i} \cong B\Sigma_k^{V_k \otimes \mathbb{C}^i}, \tag{3.3.16}$$

the Thom spectrum of the tensor product $V_k \otimes \mathbb{C}^i$ of complex vector bundles over $B\Sigma_k$. Since $V_k \otimes \mathbb{C}^i$ is a complex vector bundle, for any complex oriented cohomology theory E, we have a Thom isomorphism

$$E^*(\Sigma^{2ki} B\Sigma_k) \cong E^*(B\Sigma_k^{V_k \otimes \mathbb{C}^i}).$$

Taking $\mu_{i,k}$ to be a map representing the Thom class under this isomorphism we obtain the following diagram:

$$\begin{array}{ccc} S^{2ki} & \xrightarrow{\;\;i\;\;} & D_{\Sigma_k} S^{2i} \\ & \searrow^{\Sigma^{2ki} e} \quad \swarrow_{\mu_{i,k}} & \\ & \Sigma^{2ki} E. & \end{array}$$

Note that although the Thom classes $\mu_{i,k}$ clearly depend on the cohomology theory E, we will abuse notation and use the same symbol regardless of the cohomology theory.

When $E = MU$, McClure shows [BMMS86, VII] that the $\mu_{i,k}$ combine with the H_∞ structure maps

$$\mu_k : D_{\Sigma_k} MU \to DMU \xrightarrow{\mu} MU$$

to define an H_∞^2 structure for MU: The structure maps are those given by the top horizontal composite in Figure 3.4.

$$\begin{array}{ccccccc}
D_{\Sigma_k}(\Sigma^{2i}MU) & \longrightarrow & D_{\Sigma_k}S^{2i} \wedge D_{\Sigma_k}MU & \xrightarrow{\mu_{i,k}\wedge\mu_k} & \Sigma^{2ki}MU \wedge MU & \longrightarrow & \Sigma^{2ki}MU \\
\downarrow{\scriptstyle D_{\Sigma_k}(f)} & & \downarrow{\scriptstyle D_{\Sigma_k}S^{2i}\wedge f} & & \downarrow{\scriptstyle \Sigma^{2ki}f\wedge f} & & \downarrow{\scriptstyle \Sigma^{2ki}f} \\
D_{\Sigma_k}(\Sigma^{2i}E) & \longrightarrow & D_{\Sigma_k}S^{2i} \wedge D_{\Sigma_k}E & \xrightarrow{\mu_{i,k}\wedge\mu_k} & \Sigma^{2ki}E \wedge E & \longrightarrow & \Sigma^{2ki}E.
\end{array}$$

Figure 3.4: H^2_∞ orientations

Now consider an H_∞ complex orientation $f : MU \to E$. Figure 3.4 is induced by this structure and the left and right squares in this diagram commute for any orientation on E. The center square is the smash product of the following two squares:

$$\begin{array}{ccc}
D_{\Sigma_k}S^{2ik} & \xrightarrow{\mu_{i,k}} & \Sigma^{2ki}MU \\
\| & & \downarrow{\scriptstyle \Sigma^{2ki}f} \\
D_{\Sigma_k}S^{2ik} & \xrightarrow{\mu_{i,k}} & \Sigma^{2ki}E
\end{array}
\qquad\qquad
\begin{array}{ccc}
D_{\Sigma_k}MU & \xrightarrow{\mu_k} & MU \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma^{2ki}f} \\
D_{\Sigma_k}E & \xrightarrow{\mu_k} & E
\end{array}$$

The left square commutes since f sends MU-Thom classes to E-theory Thom classes. The right square commutes since f is an H_∞ ring map.

It follows that the center square and therefore the entire diagram commutes in Figure 3.4. Another elementary diagram chase, using the H^2_∞ structure of MU, shows that the bottom horizontal composite defines an H^2_∞ structure on E.

This gives the equivalence of H_∞ orientations and H^2_∞ orientations, since by neglect of structure every H^2_∞ orientation is H_∞.

Theorem 3.3.17. *An orientation $MU \to E$ is H_∞ if and only if it is H^2_∞.*

3.4 The Formal Group Law Perspective

3.4.1 Formal group laws

We recall some well-known facts about complex-oriented cohomology theories and formal group laws (see [Ada95, Part II] or [Rav00] for example).

Definition 3.4.1. *A (commutative, 1-dimensional) formal group law F over a commutative ring k is a connected bicommutative, associative, topological Hopf algebra $\mathcal{A}$ with a specified isomorphism $\mathcal{A} \cong k[\![x]\!]$.*

By forgetting the grading, a graded Hopf algebra of the above form is a formal group law. For such Hopf algebras the completed tensor product provides the following isomorphism:

$$\mathcal{A}\widehat{\otimes}\mathcal{A} \cong k[\![x_1, x_2]\!].$$

Notation 3.4.2. *We will frequently identify a formal group law F with the formal power series:*

$$x_1 +_F x_2 = \Delta(x) \in k[\![x_1, x_2]\!].$$

Definition 3.4.3. *Given a ring map $f : k \to k'$ and a formal group law $\mathcal{A}$ over k, the push-forward of $\mathcal{A}$ along f is the formal group law $\mathcal{A} \otimes_k^f k'$ over k'.*

One can formally define a ring L and a formal group law $\mathcal{U}$ over L such that

$$\mathcal{Ring}(L, k) \cong \text{Formal group laws over } k \tag{3.4.4}$$
$$f \to \mathcal{U} \otimes_L^f k \tag{3.4.5}$$

Theorem 3.4.6 ([Laz55])**.** *The ring L is isomorphic to a polynomial algebra over $\mathbb{Z}$ on infinitely many generators.*

Definition 3.4.7. *Given a commutative ring k we formally adjoin qth roots of unity ζ. A formal group law F over k is p-typical, if for all primes $q \neq p$, the formal sum over the qth roots of unity*

$$\sum_{\zeta^q=1}^{F} \zeta x$$

is trivial.

3.4.2 Connection to complex orientations

Recall that if X is a space and E is a spectrum, the function spectrum

$$E^X = F(\Sigma^\infty_+ X, E)$$

defines a cohomology theory satisfying

$$E^{X,*}(Y) \cong E^*(X \times Y). \tag{3.4.8}$$

Moreover, if E admits the structure of a ring spectrum (or an H_∞ ring spectrum or commutative S algebra respectively) then so does E^X.

Proposition 3.4.9 ([Lan76, 3.1])**.** *The spectra MU^{BC_p} and BP^{BC_p} are ring spectra satisfying the following natural isomorphisms:*

$$MU^{BC_p,*}X \cong MU^*(BC_p) \otimes_{MU_*} MU^*(X)$$
$$BP^{BC_p,*}X \cong BP^*(BC_p) \otimes_{BP_*} BP^*(X).$$

In complex cobordism there is a tautological element x giving an isomorphism

$$MU^*(\mathbb{C}P^\infty) \cong MU^*[\![x]\!],$$

and we fix an element ξ such that

$$MU^*(BC_p) \cong MU^*[\![\xi]\!]/[p]\xi.$$

Hence we have

$$MU^{BC_p,*}(\mathbb{C}P^\infty) \cong MU^*[\![\xi,\, x]\!]/[p]\xi.$$

An orientation $f : MU \to E$ fixes generators x and ξ in E-cohomology that define analogous isomorphisms.

The above tautological isomorphism in complex cobordism combined with the multiplication on $\mathbb{C}P^\infty$ classifying a tensor product of line bundles defines a formal group law over MU^*. An orientation $MU \to E$, induces a map $MU^* \to E^*$ which defines a formal group law structure (also denoted by E) on $E^*(\mathbb{C}P^\infty)$ by pushing forward the formal group law on MU, or equivalently [Ada95, II.4.6], by fixing the generator $x \in E^*(\mathbb{C}P^\infty)$ above.

Theorem 3.4.10 ([Qui69b])**.** *The map*

$$L \cong \mathbb{Z}[U_1, U_2, \ldots] \to MU^*$$

classifying the tautological formal group law over MU^* is an isomorphism.

Rationally, we can describe this isomorphism explicitly in terms of the cobordism classes

$$[\mathbb{C}P^n] \in MU^{-2n}.$$

Proposition 3.4.11. *There is an algebra isomorphism*

$$MU^* \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots].$$

With these choices, the power operation $P_{C_p,MU}$ is given by the following formula [Qui71]:

$$P_{C_p,MU}(x) = \prod_{i=0}^{p-1} \big([i]\xi +_{MU} x\big). \tag{3.4.12}$$

Of course, after applying an orientation $f : MU \to E$ we obtain

$$f_* P_{C_p,MU}(x) = \prod_{i=0}^{p-1} \big([i]\xi +_E x\big). \tag{3.4.13}$$

Considering Equation 3.4.13 as a power series in x whose coefficients are power series in ξ, we define

$$a_i \equiv a_i(\xi) \in E^{2(p-i-1)}(BC_p) \cong E^{2(p-i-1)}[\![\xi]\!]/[p]\xi, \quad \text{for} \quad i \geq 0$$

by the following expansion:

$$f_* P_{C_p,MU}(x) = a_0 x + a_1 x^2 + a_2 x^3 + \cdots . \tag{3.4.14}$$

By pulling back along the inclusion

$$S^2 \cong \mathbb{C}P^1 \to \mathbb{C}P^\infty,$$

and applying the C_p analogue of Equation 3.3.16 we see that a_0x is the Euler class of the regular representation of C_p and

$$a_0 = \chi, \tag{3.4.15}$$

is the Euler class of the *reduced* regular representation of C_p. Equation 3.5.12 shows that we have ring maps after inverting χ:

Proposition 3.4.16. *Let X be a topological space and define*

$$\overline{P_{C_p}} : MU^{2*}(X) \to MU^{BC_p,2*}(X)[\chi^{-1}]$$

as the map which in degree $2n$ is P_{C_p}/χ^n, then $\overline{P_{C_p}}$ and $r_*\overline{P_{C_p}}$ are maps of graded rings.

Proof. By the above discussion the result holds when X is a point. Since MU^{BC_p} and BP^{BC_p} are Landweber exact and χ is a non-zero divisor, the proposition follows from the following natural isomorphisms:

$$MU^{BC_p,*}[\chi^{-1}](X) \cong MU^*(X) \otimes_{MU^*} MU^{BC_p,*}[\chi^{-1}]$$
$$BP^{BC_p,*}[\chi^{-1}](X) \cong BP^*(X) \otimes_{BP^*} BP^{BC_p,*}[\chi^{-1}].$$

□

We now obtain formal group laws on $MU^{BC_p}[\chi^{-1}]$ and $BP^{BC_p}[\chi^{-1}]$ using the previous result and the discussion preceding Theorem 3.4.10:

$$\begin{array}{ccc} MU^{2*}(\mathbb{C}P^\infty) & \xrightarrow{\overline{P_{C_p,MU}}} & MU^{BC_p,2*}[\chi^{-1}](\mathbb{C}P^\infty) \\ \downarrow r_* & & \downarrow r_* \\ BP^{2*}(\mathbb{C}P^\infty) & \xrightarrow{\overline{P_{C_p,BP}}} & BP^{BC_p,2*}[\chi^{-1}](\mathbb{C}P^\infty) \end{array}$$

Figure 3.5: A formal group theoretic condition

Proposition 3.4.17. *The maps $\overline{P_{C_p,MU}}$ and $r \circ \overline{P_{C_p,MU}}$ in Figure 3.5 define formal group laws $\mathcal{UP}$ and $\mathcal{VP}$ over $MU^{BC_p}[\chi^{-1}]$ and $BP^{BC_p}[\chi^{-1}]$ respectively.*

Theorem 3.4.18. *The map $r : MU \to BP$ is a map of H_∞ ring spectra if and only if $\mathcal{VP}$ is p-typical.*

Proof. Since the map r is a p-universal orientation of BP, there exists a map

$$P : BP \to BP^{BC_p}[\chi^{-1}].$$

that makes Figure 3.5 commute if and only if $\mathcal{VP}$ is p-typical. This happens if and only if the polynomial generators in MU^{-2n} map to zero under $\overline{P_{C_p,MU}}$ when $n \neq p^i - 1$. Since the cobordism classes $[\mathbb{C}P^n]$ are rationally polynomial generators and all rings in sight are torsion-free, we see that $\mathcal{VP}$ is p-typical if and only if the elements MC_n described in Step 6 map to 0. □

3.5 Computing the Obstructions

Before proving Proposition 3.5.21 we will need some notation.

3.5.1 Notation

Throughout this paper, the symbol

$$\alpha = (\alpha_0, \alpha_1, \dots) \tag{3.5.1}$$

with $\alpha_n = 0$ for $n \gg 0$, will be a multi-index beginning with α_0.

As the reader will see, it will also be convenient to have notation for multi-indices starting with α_1, so we let

$$\overline{\alpha} = (\alpha_1, \alpha_2, \dots). \tag{3.5.2}$$

Given an infinite list of variables $a_0, a_1, a_2, \dots$, we set

$$a^\alpha = a_0^{\alpha_0} a_1^{\alpha_1} \dots \quad \text{and} \quad a^{\overline{\alpha}} = a_1^{\alpha_1} a_2^{\alpha_2} \dots. \tag{3.5.3}$$

For any integer n we define the modified multinomial coefficient $\mu(n;\overline{\alpha})$ by the formal power series expansion:

$$(1+b_1+b_2\cdots)^n = \sum_{\overline{\alpha}} \mu(n;\overline{\alpha}) b^{\overline{\alpha}}. \tag{3.5.4}$$

We also set:

$$|\alpha| = \sum_{i\geq 0} \alpha_i \tag{3.5.5}$$

$$|\alpha|' = \sum_{i\geq 0} i\alpha_i = |\overline{\alpha}|'. \tag{3.5.6}$$

Given a formal power series $S(z)$, let

$$S(z)[z^k] = \text{coefficient of } z^k \text{ in } S(z). \tag{3.5.7}$$

3.5.2 Additive and multiplicative operations

Recall that the Landweber-Novikov algebra is the subalgebra of MU^*MU whose elements define additive cohomology operations. This algebra is a free $\mathbb{Z}_{(p)}$-module on elements

$$s_{\alpha_1,\alpha_2,\ldots} = s_{\overline{\alpha}} \tag{3.5.8}$$

dual to the standard basis

$$t_1^{\alpha_1} t_2^{\alpha_2} \cdots = t^{\overline{\alpha}} \in MU_{2|\overline{\alpha}|'}MU \cong MU_{2|\overline{\alpha}|'}BU. \tag{3.5.9}$$

To simplify our formulas we extend the indexing to multi-indices starting with α_0 by setting

$$s_\alpha \equiv s_{\overline{\alpha}} \in MU^{2|\alpha|'}MU. \tag{3.5.10}$$

Theorem 3.5.11 ([Qui71])**.** *If $x \in MU^{-2q}(X)$ and $m \gg 0$ then*

$$\chi^{m+q} P_{C_p} x = \sum_{|\alpha|=m} a^\alpha s_\alpha(x). \tag{3.5.12}$$

Since the right hand side of Equation 3.5.12 is additive in x and P_{C_p} is always multiplicative, we obtain Proposition 3.4.16 by inverting χ.

For any complex oriented cohomology theory E,

$$[i]\xi +_E x \equiv i\xi \mod x,$$

which implies

$$\chi = a_0 \equiv (p-1)!\xi^{p-1} \mod \xi^p. \tag{3.5.13}$$

It follows that inverting χ factors through inverting ξ, so when E is MU or BP, we have:

$$E^{BC_p,*}(X)[\chi^{-1}] \cong E^*(X)[\![\xi]\!][\chi^{-1}]/[p]\xi \cong E^*(X)[\![\xi]\!][\chi^{-1}]/\langle p\rangle\xi.$$

Since

$$\langle p\rangle\xi = [p]\xi/\xi \equiv p \mod \xi$$

and $(p-1)!$ is not divisible by p, $q_*\chi$ is not a zero-divisor in $MU^*[\![\xi]\!]/\langle p\rangle\xi$ and similarly for $r_*q_*\chi$. It follows that, when $E = MU$ or BP, the localization map

$$E^*(X)[\![\xi]\!]/\langle p\rangle\xi \to E^*(X)[\![\xi]\!][\chi^{-1}]/\langle p\rangle\xi$$

is an injection. Applying Proposition 3.4.16 proves the following:

Proposition 3.5.14. *The composites*

$$\begin{aligned} q_*P_{C_p} :& MU^*(\mathbb{C}P^\infty) \to MU^{BC_p,*}(\mathbb{C}P^\infty)/\langle p\rangle\xi \\ r_*q_*P_{C_p} :& MU^*(\mathbb{C}P^\infty) \to BP^{BC_p,*}(\mathbb{C}P^\infty)/\langle p\rangle\xi \end{aligned}$$

are ring maps.

3.5.3 Derivation of MC_n

The following result shows that if $x = [\mathbb{C}P^n]$ in Equation 3.5.12 we can take $m = n$:

Lemma 3.5.15.

$$\chi^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} a^{\alpha} s_{\alpha}[\mathbb{C}P^n]. \tag{3.5.16}$$

Proof. By Equation 3.5.12, for $k \gg 0$ we have:

$$\begin{aligned}
\chi^{2n+k} P_{C_p}[\mathbb{C}P^n] &= \sum_{|\alpha|=n+k} a^{\alpha} s_{\alpha}[\mathbb{C}P^n] \\
&= \sum_{\alpha_0=0}^{n+k} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a_0^{\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= \sum_{\alpha_0=0}^{k-1} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] + \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n]
\end{aligned}$$

Since MU^* is concentrated in non-positive degrees,

$$s_{\overline{\alpha}}([\mathbb{C}P^n]) \in MU^{2|\overline{\alpha}|'-2n} = 0$$

when $|\overline{\alpha}|' > n$.

In the first sum of the last equation, $|\overline{\alpha}| > n$. Since

$$|\overline{\alpha}|' = \sum_{i\geq 1} i\alpha_i \geq \sum_{i \geq 1} \alpha_i = |\overline{\alpha}|,$$

all terms in the first sum are trivial.

We are left with

$$\begin{aligned}
\chi^{2n+k} P_{C_p}[\mathbb{C}P^n] &= \sum_{\alpha_0=k}^{n+k} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n+k-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= a_0^k \sum_{\alpha_0=0}^{n} a_0^{\alpha_0} \sum_{|\overline{\alpha}|=n-\alpha_0} a^{\overline{\alpha}} s_{\overline{\alpha}}[\mathbb{C}P^n] \\
&= a_0^k \sum_{|\alpha|=n} a^{\alpha} s_{\alpha}[\mathbb{C}P^n]
\end{aligned}$$

Since $a_0 = \chi$ is a not a zero-divisor the lemma follows. □

Theorem 3.5.17 ([Ada95, I.8.1])**.**

$$s_\alpha[\mathbb{C}P^n] = \mu(-(n+1);\overline{\alpha})[\mathbb{C}P^{n-|\alpha|'}] \tag{3.5.18}$$

We combine Equations 3.5.16 and 3.5.18 and obtain:

Theorem 3.5.19.

$$MC_n(\xi) \equiv r_* q_* \chi^{2n} P_{C_p}[\mathbb{C}P^n] = \sum_{|\alpha|=n} \mu(-(n+1);\overline{\alpha})\ r_*[\mathbb{C}P^{n-|\alpha|'}]\ a^\alpha.$$

Remark 3.5.20. After correcting a couple of typographical errors, this is a simplified version of the formula given in [BMMS86, VIII.7.8].

For $n \neq p^k - 1$, the power series $MC_n(\xi)$ are McClure's obstructions to the existence of H_∞ strucure on Quillen's map $r : MU \to BP$. Note that, if $i+1$ is not a power of p then $r_*[\mathbb{C}P^i] = 0$, so many of the summands on MC_n are zero. For our calculations, we make use of the following alternate expression:

Proposition 3.5.21. *McClure's formula is equivalent to*

$$MC_n(\xi) = \chi^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} [z^k].$$

Proof. We rearrange the sum by summing over $|\alpha|' = k$. Now the condition $|\alpha| = n$ is simply a constraint on α_0.

$$\begin{aligned} MC_n(\xi) &= \sum_{k=0}^{n} \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}} \mu(-(n+1);\overline{\alpha})\ r_*[\mathbb{C}P^{n-|\alpha|'}]\ a^\alpha \\ &= \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}} \mu(-(n+1);\overline{\alpha})\ a^\alpha. \end{aligned}$$

To simplify the inner sum, we consider the following formal series and use the definition of the modified multinomial coefficients given in Equation 3.5.4:

$$\begin{aligned}
a_0^{2n+1}\left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)} &= a_0^n\left(1+\frac{a_1}{a_0}z+\frac{a_2}{a_0}z^2+\cdots\right)^{-(n+1)} \\
&= a_0^n\sum_{\overline{\alpha}}\mu(-(n+1);\overline{\alpha})\left(\frac{a_1}{a_0}z\right)^{\alpha_1}\left(\frac{a_2}{a_0}z^2\right)^{\alpha_2}\cdots \\
&= \sum_{\overline{\alpha}}\mu(-(n+1);\overline{\alpha})\,\frac{a_0^n\, a_1^{\alpha_1} a_2^{\alpha_2}\cdots}{a_0^{\alpha_1+\alpha_2+\cdots}}\, z^{\alpha_1+2\alpha_2+\cdots} \\
&= \sum_{k\geq 0} z^k\left(\sum_{|\overline{\alpha}|'=k}\mu(-(n+1);\overline{\alpha})\,\frac{a_0^n\, a_1^{\alpha_1}\, a_2^{\alpha_2}\cdots}{a_0^{\alpha_1+\alpha_2+\cdots}}\right)
\end{aligned}$$

Now we consider the coefficients of z^k. For $k \leq n$, the restriction $|\overline{\alpha}|' = k$ implies $|\overline{\alpha}| \leq n$. Hence we may extend to a sum over multi-indices $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ with $\alpha_0 = n - |\overline{\alpha}|$ which forces $|\alpha| = n$. Thus we have, for $0 \leq k \leq n$,

$$a_0^{2n+1}\left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-(n+1)}[z^k] = \sum_{\substack{|\alpha|'=k \\ |\alpha|=n}}\mu(-(n+1);\overline{\alpha})\, a^{\alpha}.$$

□

3.5.4 Sparseness

In this section we prove that, at odd primes, many of the MC_n do in fact vanish. We also give a sparseness result for the a_i.

Proposition 3.5.22. *If* $n \not\equiv 0 \mod p-1$ *then* $MC_n = 0$.

Proof. The statement is vacuously true at the prime 2, so we will assume p is odd. The summands of the equation in Theorem 3.5.19 are constant multiples of $r_*[\mathbb{C}P^i]$ and a^α. The first term is nonzero only in degrees divisible by $2(p-1)$ and it follows from the lemma below that the nonzero a^α are also concentrated in degrees divisible by $2(p-1)$.

Now the left side of the equation in Theorem 3.5.19 is in degree $2n(p-2)$ and the right hand side is concentrated in degrees divisible by $2(p-1)$. Since 2 and $(p-2)$ are units mod p we see that MC_n can only be non-zero when n is divisible by $p-1$. □

Lemma 3.5.23. *The elements $a_i \in BP^*(BC_p)$ defined in Equation 3.4.14 are zero if $i \not\equiv 0$* mod $p-1$.

Proof. Since the lemma is vacuously true for $p = 2$, we will assume p is odd.

The action of $C_p^\times$ on C_p induces an action of $C_p^\times$ on BC_p. In $BP^*(BC_p)$, an element $v \in C_p^\times$ acts on $[i]\xi$ by

$$[i]\xi \mapsto [vi]\xi.$$

Since the product

$$\prod_{i=1}^{p-1}([i]\xi +_{BP} x)$$

is invariant under this action, we see that $a_i \in BP^{2(p-i-1)}(BC_p)^{C_p^\times}$.

The Atiyah-Hirzebruch spectral sequence computing $BP^*(BC_p)$ collapses at the E_2 page, which is of the form $H^*(BC_p, BP^*)$. The group action above induces a group action on this page. Since the edge homomorphism $BP^*(BC_p) \to H^*(BC_p)$, is an equivariant surjection that restricts to an isomorphism along the 0th row, the associated graded of $BP^*(BC_p)^{C_p^\times}$ is isomorphic to $H^*(BC_p)^{C_p^\times} \otimes BP^* \cong \mathbb{Z}/p[\xi^{p-1}] \otimes BP^*$.

Since this last group is concentrated in degrees divisible by $2(p-1)$, if $a_i \neq 0$ then

$$a_i \in BP^{2(p-1)*}(BC_p).$$

The congruence

$$\frac{|a_i|}{2} = (p-1-i) \equiv i \equiv 0 \mod (p-1)$$

implies i is divisible by $p-1$.

□

As a result, it is of interest to check $MC_{2(p-1)}$. In this case, one can give the formula more explicitly:

$$\begin{aligned} MC_{2(p-1)}(\xi) = a_0^{2p-4} r_*[\mathbb{C}P^{(p-1)}] \left(-(2p-1)a_0 a_{(p-1)}\right) \\ + a_0^{2p-4} r_*[\mathbb{C}P^0] \left(-(2p-1)a_0 a_{2(p-1)} + p(2p-1)a_{(p-1)}^2\right) \end{aligned} \tag{3.5.24}$$

Making the simplifications $[\mathbb{C}P^0] = 1$ and $r_*[\mathbb{C}P^{p-1}] = v_1$, we have

$$MC_{2(p-1)}(\xi) = (2p-1)a_0^{2p-4} \left(-v_1 a_0 a_{(p-1)} - a_0 a_{2(p-1)} + p a_{(p-1)}^2\right)$$

3.6 Calculations

In this section, we outline the computation of the MC_n, work through an example at the prime 2, and display results at the primes 2, 3, and 5. We have developed a Mathematica package [JN09a] to automate the calculations, together with a user's guide [JN09b].

3.6.1 Description of calculation

We are working in $BP^*[\![\xi]\!]/\langle p\rangle\xi$, and we emphasize reduction modulo $\langle p\rangle\xi$ by writing $\equiv \bmod \langle p\rangle\xi$ instead of equality. Our calculations have three parameters: the prime, p, the value of n, and a truncation number, k. All of our computations are modulo $(\xi, x)^{k+1}$. If power series $f(\xi)$ and $g(\xi)$ are equal modulo the ideal $(\xi)^{k+1}$, we write

$$f(\xi) = g(\xi) + O(\xi)^{k+1}.$$

It is important to note, because of this choice, that the range of accurate coefficients for the $a_i(\xi)$ decreases as i grows. Each a_i is accurate modulo ξ^{k-i+1}. Using the formula above, and the fact that $a_0 = (p-1)! \cdot \xi^{p-1} + \cdots$, we see that $MC_{2(p-1)}$ is accurate modulo ξ^{k-p+2}.

Our package represents power series as coefficient lists, with the length of the list determined by the truncation number, k. Our implementation of multiplication and composition of series as operations on these lists preserves this accuracy, without computing unnecessary terms. We have made other efforts to streamline the computation, but our results are limited by the difficulty of formal group law calculations. Determining the series $\exp_{BP}$ is already a task whose computation time grows quickly with the length of the input. Calculating the a_i is also a high-complexity task, and as a result we do not expect direct computation to be a feasable approach for large primes. We have not been able to work in a large enough range to detect non-zero values of MC_n for primes greater than 3.

3.6.2 Example calculation, $p = 2$

To give the reader a sense of how these calculations are implemented, we work through the calculation of $MC_2(\xi)$ with the minimum range of coefficients necessary to see that it is non-zero. For this, it is necessary to work modulo $(x, \xi)^8$. The formula for MC_2 is given in Proposition 3.5.21:

$$MC_2(\xi) = a_0^5 \sum_{k=0}^{2} r_*[\mathbb{C}P^{n-k}] \cdot \left(\textstyle\sum_{i \geq 0} a_i z^i\right)^{-(n+1)} [z^k].$$

Now one can easily check the formal computation

$$\begin{aligned}\left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-1} &= a_0^{-1} - a_1 a_0^{-2} z + \left(-a_2 a_0^{-2} + a_1^2 a_0^{-3}\right) z^2 \\ &\quad + O(z)^3\end{aligned}$$

and hence

$$\begin{aligned}\left(\textstyle\sum_{i\geq 0} a_i z^i\right)^{-3} &= a_0^{-3} - 3a_1 a_0^{-4} z + \left(-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5}\right) z^2 \\ &\quad + O(z)^3.\end{aligned}$$

The image of $[\mathbb{C}P^i] \in MU^{-2i}$ under r_* is given by

$$r_*[\mathbb{C}P^i] = \begin{cases} 0 & \text{if } i \neq p^k - 1 \\ [\mathbb{C}P^i] = p^k \ell_k & \text{if } i = p^k - 1 \end{cases}$$

The elements ℓ_k are rational generators for BP, but it is convenient to work with integral generators. For this example we choose the Hazewinkel generators v_k, but the result is independent of this choice. It will be necessary only to use $v_1 = 2\ell_1$, so we work modulo the ideal $I = (v_2, v_3, \ldots)$. Modulo I we have $4\ell_2 = v_1^3$, and this will be the only additional substitution we need to use.

Returning to the calculation, we have

$$[\mathbb{C}P^0] = 1, \quad r_*[\mathbb{C}P^1] = 2\ell_1 = v_1, \quad \text{and } r_*[\mathbb{C}P^2] = 0$$

and so

$$\begin{aligned} MC_2(\xi) &= a_0^5 \left(-3v_1 a_0^{-4} a_1 + (-3a_2 a_0^{-4} + 6a_1^2 a_0^{-5})\right) \\ &= 6a_1^2 - 3a_0 a_2 - 3v_1 a_0 a_1. \end{aligned}$$

To continue, we determine $a_0(\xi)$, $a_1(\xi)$, and $a_2(\xi)$. These are defined by the following (3.4.13 3.4.14):

$$\begin{aligned} P_{C_p,BP}(x) = r_* P_{C_p,MU}(x) = \prod_{i=0}^{1} ([i]\xi +_{BP} x) &= x \cdot \exp\left(\log(\xi) + \log(x)\right) \\ = x \cdot \Big[a_0 + a_1 x^1 + a_2 x^2 + a_3 x^3 & \\ + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 & \\ + O(x, \xi)^8 \Big]. & \end{aligned}$$

The logarithm is

$$\log_{BP}(\xi) = \xi + \ell_1 \xi^2 + \ell_2 \xi^4 + O(\xi)^8$$

and hence the exponential is

$$\begin{aligned}\exp_{BP}(\xi) = \xi - \ell_1\xi^2 + 2\ell_1^2\xi^3 + \left(-5\ell_1^3 - \ell_2\right)\xi^4 \\ + \left(14\ell_1^4 + 6\ell_1\ell_2\right)\xi^5 \\ + \left(-42\ell_1^5 - 28\ell_1^2\ell_2\right)\xi^6 \\ + \left(132\ell_1^6 + 120\ell_1^3\ell_2 + 4\ell_2^2\right)\xi^7 \\ + O(\xi)^8.\end{aligned}$$

Using the logarithm and exponential, we give the reduced 2-series:

$$\begin{aligned}\langle 2\rangle\xi = \frac{1}{\xi}\exp(2\log(\xi)) = 2 - 2\ell_1\xi + 8\ell_1^2\xi^2 \\ + \left(-36\ell_1^3 - 14\ell_2\right)\xi^3 \\ + \left(176\ell_1^4 + 120\ell_1\ell_2\right)\xi^4 \\ + \left(-912\ell_1^5 - 888\ell_1^2\ell_2\right)\xi^5 \\ + \left(4928\ell_1^6 + 6240\ell_1^3\ell_2 + 448\ell_2^2\right)\xi^6 \\ + O(\xi)^7\end{aligned}$$

Substituting the Hazewinkel generators, and working modulo v_2,

$$\begin{aligned}\langle 2\rangle\xi = 2 - v_1\xi + 2v_1^2\xi^2 \\ - 8v_1^3\xi^3 \\ + 26v_1^4\xi^4 \\ - 84v_1^5\xi^5 \\ + 300v_1^6\xi^6 \\ + O(\xi)^7\end{aligned}$$

and

$$
\begin{aligned}
P_{C_p,BP} = x \cdot \Big[& \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big) \\
& - \ell_1 \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^2 \\
& + 2\ell_1^2 \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^3 \\
& + \Big(-5\ell_1^3 - \ell_2\Big) \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^4 \\
& + \Big(14\ell_1^4 + 6\ell_1\ell_2\Big) \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^5 \\
& + \Big(-42\ell_1^5 - 28\ell_1^2\ell_2\Big) \cdot \\
& \qquad \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^6 \\
& + \Big(132\ell_1^6 + 120\ell_1^3\ell_2 + 4\ell_2^2\Big) \cdot \\
& \qquad \Big(\xi + x + \ell_1(\xi^2 + x^2) + \ell_2(\xi^4 + x^4) + \ell_3(\xi^8 + x^8)\Big)^7 \\
& + O(x,\xi)^8 \Big].
\end{aligned}
$$

Expanding, and substituting the Hazewinkel generators, we have

$$
\begin{aligned}
a_0 &= \xi + O(\xi)^8 \\
a_1 &= 1 - v_1\xi + v_1^2\xi^2 - 2v_1^3\xi^3 \\
&\quad + 3v_1^4\xi^4 - 4v_1^5\xi^5 \\
&\quad + v_1^6\xi^6 + O(\xi)^7 \\
&\equiv 1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7 \quad \bmod \langle 2\rangle\xi \\
a_2 &= v_1^2\xi - 4v_1^3\xi^2 + 10v_1^4\xi^3 - 21v_1^5\xi^4 \\
&\quad + 43v_1^6\xi^5 + O(\xi)^6 \\
&\equiv v_1^2\xi + v_1^5\xi^4 + O(\xi)^6 \quad \bmod \langle 2\rangle\xi
\end{aligned}
$$

Substituting into the formula for MC_2, we have (modulo v_2)

$$\begin{aligned}MC_2(\xi) =& 6a_1^2 - a_0a_2 - 3v_1a_0a_1 \\ \equiv& 6\left(1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7\right)^2 \\ &- 3\left(\xi + O(\xi)^8\right)\left(v_1^2\xi + v_1^5\xi^4 + O(\xi)^6\right) \\ &- 3v_1\left(\xi + O(\xi)^8\right)\left(1 + v_1\xi + v_1^4\xi^4 + v_1^5\xi^5 + v_1^6\xi^6 + O(\xi)^7\right) \\ &\quad \bmod \langle 2\rangle\xi \\ =& 6 + 9v_1\xi + 12v_1^4\xi^4 + 18v_1^5\xi^5 + 21v_1^6\xi^6 + O(\xi)^7 \quad \bmod \langle 2\rangle\xi.\end{aligned}$$

Note that, although a_2 is accurate only modulo ξ^6, the product a_0a_2 is accurate modulo ξ^7 and hence MC_2 is accurate modulo ξ^7. Since the lowest-order term is $3 \cdot 2$, we subtract $3 \cdot \langle 2\rangle\xi$ to give

$$MC_2(\xi) \equiv 12v_1\xi - 6v_1^2\xi^2 + v_1^3\xi^3 - 66v_1^4\xi^4 + 270v_1^5\xi^5 - 879v_1^6\xi^6 + O(\xi)^7 \quad \bmod \langle 2\rangle\xi.$$

Continuing to reduce in this way gives the following:

$$MC_2(\xi) \equiv\ v_1^6\xi^6 + O(\xi)^7 \qquad \bmod \langle 2\rangle\xi.$$

Since the lowest-order term of the right-hand side is non-zero mod 2, the entire expression is non-zero in $BP^*[\![\xi]\!]/\langle 2\rangle\xi$.

3.6.3 Results at $p = 2$

$$
\begin{aligned}
\langle 2\rangle\xi = 2 &- \xi v_1 + 2\xi^2 v_1^2 + \xi^3\left(-8v_1^3 - 7v_2\right) + \xi^4\left(26v_1^4 + 30v_1v_2\right)\\
&+ \xi^5\left(-84v_1^5 - 111v_1^2v_2\right) + \xi^6\left(300v_1^6 + 502v_1^3v_2 + 112v_2^2\right)\\
&+ \xi^7\left(-1140v_1^7 - 2299v_1^4v_2 - 960v_1v_2^2 - 127v_3\right)\\
&+ \xi^8\left(4334v_1^8 + 9958v_1^5v_2 + 5414v_1^2v_2^2 + 766v_1v_3\right)\\
&+ \xi^9\Big(-16692v_1^9 - 43118v_1^6v_2 - 29579v_1^3v_2^2 - 2380v_2^3\\
&\qquad -3579v_1^2v_3\Big)\\
&+ \xi^{10}\Big(65744v_1^{10} + 189976v_1^7v_2 + 161034v_1^4v_2^2\\
&\qquad +31012v_1v_2^3 + 17770v_1^3v_3 + 5616v_2v_3\Big)\\
&+ \xi^{11}\Big(-262400v_1^{11} - 837637v_1^8v_2 - 838452v_1^5v_2^2\\
&\qquad -240631v_1^2v_2^3 - 86487v_1^4v_3 - 55329v_1v_2v_3\Big)\\
&+ \xi^{12}\Big(1056540v_1^{12} + 3685550v_1^9v_2 + 4232750v_1^6v_2^2\\
&\qquad +1600786v_1^3v_2^3 + 58268v_2^4\\
&\qquad +404198v_1^5v_3 + 363210v_1^2v_2v_3\Big)\\
&+ \xi^{13}\Big(-4292816v_1^{13} - 16254540v_1^{10}v_2\\
&\qquad - 21110372v_1^7v_2^2 - 10071369v_1^4v_2^3\\
&\qquad - 1022466v_1v_2^4 - 1864478v_1^6v_3\\
&\qquad -2193009v_1^3v_2v_3 - 212440v_2^2v_3\Big)\\
&+ O(\xi)^{14}
\end{aligned}
$$

$$
\begin{aligned}
MC_1(\xi) \equiv\ & \xi^2 v_1^2 + \xi^3 v_2 + \xi^4 \left(v_1^4 + v_1 v_2\right) \\
& + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^8 + v_1 v_3\right) \\
& + \xi^9 \left(v_1^9 + v_1^6 v_2 + v_1^3 v_2^2 + v_2^3 + v_1^2 v_3\right) \\
& + \xi^{10} \left(v_1^{10} + v_1 v_2^3 + v_1^3 v_3\right) \\
& + \xi^{11} \left(v_1^5 v_2^2 + v_1 v_2 v_3\right) \\
& + \xi^{12} \left(v_1^{12} + v_1^9 v_2 + v_1^6 v_2^2 + v_1^3 v_2^3 \right. \\
& \qquad \left. + v_2^4 + v_1^5 v_3\right) \\
& + \xi^{13} v_1^4 v_2^3 \\
& + O(\xi)^{14} \mod \langle 2 \rangle \xi
\end{aligned}
$$

$$
\begin{aligned}
MC_2(\xi) \equiv\ & \xi^6 \left(v_1^6 + v_2^2\right) + \xi^7 \left(v_1^7 + v_3\right) + \xi^8 \left(v_1^5 v_2 + v_1 v_3\right) \\
& + \xi^9 v_2^3 + \xi^{10} \left(v_1^4 v_2^2 + v_1 v_2^3\right) + \xi^{11} \left(v_1^5 v_2^2 + v_1^2 v_2^3 + v_1^4 v_3\right) \\
& + \xi^{12} \left(v_1^9 v_2 + v_1^5 v_3\right) + \xi^{13} \left(v_1^{13} + v_1^{10} v_2 + v_1^3 v_2 v_3\right) \\
& + O(\xi)^{14} \mod \langle 2 \rangle \xi
\end{aligned}
$$

$$
\begin{aligned}
MC_3(\xi) \equiv\ & \xi^6 v_1^6 + \xi^7 \left(v_1^4 v_2 + v_1 v_2^2\right) + \xi^8 \left(v_1^8 + v_1^5 v_2 + v_1 v_3\right) \\
& + \xi^{10} \left(v_1^{10} + v_1^7 v_2 + v_1^4 v_2^2 + v_1^3 v_3 + v_2 v_3\right) \\
& + \xi^{11} \left(v_1^{11} + v_1^8 v_2 + v_1^4 v_3 + v_1 v_2 v_3\right) \\
& + \xi^{12} v_1^3 v_2^3 \\
& + \xi^{13} \left(v_1^{13} + v_1^3 v_2 v_3 + v_2^2 v_3\right) \\
& + O(\xi)^{14} \mod \langle 2 \rangle \xi
\end{aligned}
$$

$$
\begin{aligned}
MC_4(\xi) \equiv\ & \xi^{10} v_1^4 v_2^2 + \xi^{11} \left(v_1^{11} + v_1^8 v_2 + v_1^5 v_2^2 + v_1^4 v_3 \right) \\
& + \xi^{12} \left(v_1^9 v_2 + v_1^3 v_2^3 + v_2^4 \right) \\
& + \xi^{13} \left(v_1^{10} v_2 + v_1^4 v_2^3 + v_1^6 v_3 + v_1^3 v_2 v_3 + v_2^2 v_3 \right) \\
& + O(\xi)^{14} \quad \mathrm{mod}\ \langle 2 \rangle \xi
\end{aligned}
$$

$$
MC_5(\xi) \equiv 0 + O(\xi)^{14} \quad \mathrm{mod}\ \langle 2 \rangle \xi
$$

For $n > 5$, we have not found any non-zero MC_n mod $(\xi^{14}, \langle 2 \rangle \xi)$.

3.6.4 Results at $p = 3$

$$
\begin{aligned}
\langle 3\rangle\xi = 3 &- 8\xi^2 v_1 + 72\xi^4 v_1^2 - 840\xi^6 v_1^3 \\
&+ \xi^8 \left(9000 v_1^4 - 6560 v_2\right) \\
&+ \xi^{10} \left(-88992 v_1^5 + 216504 v_1 v_2\right) \\
&+ \xi^{12} \left(658776 v_1^6 - 5360208 v_1^2 v_2\right) \\
&+ \xi^{14} \left(1199088 v_1^7 + 119105576 v_1^3 v_2\right) \\
&+ \xi^{16} \left(-199267992 v_1^8 - 2424100032 v_1^4 v_2 \right.\\
&\qquad \left. +129120480 v_2^2\right) \\
&+ \xi^{18} \left(5896183992 v_1^9 + 45824243688 v_1^5 v_2 \right.\\
&\qquad \left. -8307203592 v_1 v_2^2\right) \\
&+ \xi^{20} \left(-133449348816 v_1^{10} - 807801733088 v_1^6 v_2 \right.\\
&\qquad \left. +336744805688 v_1^2 v_2^2\right) \\
&+ \xi^{22} \left(2658275605728 v_1^{11} + 13162584394728 v_1^7 v_2 \right.\\
&\qquad \left. -11021856839856 v_1^3 v_2^2\right) \\
&+ \xi^{24} \left(-48579725371464 v_1^{12} - 193206868503840 v_1^8 v_2 \right.\\
&\qquad \left. +314960186505360 v_1^4 v_2^2 - 3670852206240 v_2^3\right) \\
&+ O(\xi)^{26}
\end{aligned}
$$

$$
\begin{aligned}
MC_2(\xi) \equiv\ & v_1^3\xi^8 + 2v_2\xi^{10} + \left(v_1^5 + v_2v_1\right)\xi^{12} \\
& + 2v_1^2v_2\xi^{14} + 2v_1^7\xi^{16} + \left(2v_1^8 + v_2^2\right)\xi^{18} \\
& + \left(v_2v_1^5 + v_2^2v_1\right)\xi^{20} \\
& + \left(2v_1^{10} + 2v_2v_1^6 + v_2^2v_1^2\right)\xi^{22} \\
& + \left(v_1^{11} + v_2v_1^7\right)\xi^{24} \\
& + O(\xi)^{26} \qquad \text{mod } \langle 3\rangle\xi
\end{aligned}
$$

$$
MC_4(\xi) \equiv 2v_1^9\xi^{22} + 2v_1^{10}\xi^{24} + O(\xi)^{26} \qquad \text{mod } \langle 3\rangle\xi
$$

For $n > 4$, we have not found any non-zero MC_n mod $(\xi^{26}, \langle 3\rangle\xi)$.

3.6.5 Results at $p = 5$

$$\begin{aligned}\langle 5\rangle\xi = 5 &- 624\xi^4 v_1 + 390000\xi^8 v_1^2 - 341094000\xi^{12} v_1^3 \\ &+ 347012281200\xi^{16} v_1^4 \\ &- 384865568096880\xi^{20} v_1^5 \\ &+ 439473937694610000\xi^{24} v_1^6 \\ &- 505939291320703500000\xi^{28} v_1^7 \\ &+ 580757413801495562502000\xi^{32} v_1^8 \\ &+ O(\xi)^{40}\end{aligned}$$

$$MC_4(\xi) \equiv \xi^{32} v_1^5 + 4\xi^{36} v_2 + O(\xi)^{40} \mod \langle 5\rangle\xi$$

For $n > 4$, we have not found any non-zero MC_n mod $(\xi^{40}, \langle 5\rangle\xi)$.

CHAPTER 4

$H_\infty \neq E_\infty$

4.1 Introduction

We have too many high sounding words, and too few actions that correspond to them.

- Abigail Adams

This note is based on the following theorem of Kraines-Lada:

Theorem 4.1.1 ([KL79])**.** *Let s be a generator of $\mathrm{Prim} H^{30}(BU;\mathbb{Z}_{(2)})$. Define KL by the following fibration sequence:*

$$KL \xrightarrow{i} BU_{(2)} \xrightarrow{4s} K(\mathbb{Z}_{(2)}, 30).$$

Then i is a map of $L(2)$ spaces, but the $L(2)$ structure on KL does not lift to an E_3 structure. In particular, KL does not admit an E_∞ structure compatible with this $L(2)$ structure.

After some translation we will prove:

Theorem 4.1.2. *The map*

$$\Sigma^\infty_+ KL \xrightarrow{\Sigma^\infty_+ i} \Sigma^\infty_+ BU_{(2)}$$

is a map of H_∞ ring spectra, but the H_∞ ring structure on $\Sigma^\infty_+ KL$ does not lift to an E_3 structure. In particular, $\Sigma^\infty_+ KL$ does not admit a compatible E_∞ ring structure.

This theorem follows immediately from the following definitions and lemmas from [May09]:

4.2 $L(n)$ spaces and spectra

Notation 4.2.1. *Let $\mathcal{L}$ be the linear isometries operad. We will use L to denote the associated reduced monad on pointed spaces with Cartesian products, spaces under S^0 with smash products, and spectra under S^0 with smash products.*

In particular:

- L is an endofunctor on pointed spaces satisfying

$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$

  where $\sim$ represents the obvious base point identifications.

- L is an endofunctor on spaces under S^0 satisfying

$$LY = \coprod_{n \geq 0} \mathcal{L}(n) \times_{\Sigma_n} Y^n/(\sim),$$

  where $\sim$ represents the obvious unit map identifications.

- L is an endofunctor on the Lewis-May-Steinberger category of spectra (see [LMS86]) under S^0 satisfying

$$LE = \bigvee_{n \geq 0} \mathcal{L}(n) \ltimes_{\Sigma_n} E^{\wedge n}/(\sim),$$

  where $\sim$ represents the obvious unit map identifications (see [EKMM97, 4.9,6.1]).

We justify this abuse of notation with the following lemma:

Lemma 4.2.2 ([May09, 4.8, p. 1027])**.** *We have the following chain of homeomorphisms* *natural in based spaces*[1] X

$$\begin{aligned}\Sigma^\infty_+ LX &\equiv \Sigma^\infty (LX)_+ \\ &\cong \Sigma^\infty L(X_+) \\ &\cong L\Sigma^\infty X_+ \\ &\equiv L\Sigma^\infty_+ X.\end{aligned}$$

For simplicity, for the remainder of this paper we will assume all spaces are non-degenerately based.

Recall that the category of L-algebras in group-like pointed spaces is equivalent to the category of infinite loop spaces. The following definition provides a categorical filtration between spaces and homotopy coherent L-algebras (which are weakly equivalent to L-algebras).

Definition 4.2.3. *A based space X is $L(n)$ if one can construct the $n-1$ skeleton (in the simplicial direction) of the augmented simplicial space*

$$B(L, L, X) \xrightarrow{\mu} X,$$

such that the canonical map

$$X \to LX \hookrightarrow B(L, L, X)$$

is a section of μ.

Remark 4.2.4. Despite the similarity in notation, we remind the reader that the *property* of being $L(n)$ has nothing to do with the *space* $\mathcal{L}(n)$.

Remark 4.2.5. Note that our definition of a $L(n)$ space is different from that of a Q_n space used in [KL79]. Kraines and Lada restrict to the case when X is connected, in which case L could be replaced with $Q = \Omega^\infty \Sigma^\infty$. In this respect, our definition is more general.

1. It is helpful to think of this basepoint as a multiplicative unit.

For reasons that are unclear to the author, their definition of a Q_n space is a cubical analogue of the above definition, while maps of Q_n spaces are defined simplicially. Such a definition requires one to continually translate between these two worlds. We take this opportunity to propose the above alternative definition which is simpler to manipulate and can be easily adapted to any reasonable category of algebras over an operad.

Restricting to connected spaces, one can probably relate the two definitions using the Quillen equivalence between simplicial and cubical sets [Jar06, Cis06]. In any case, we only require these notions to coincide when X is connected and $n \leq 2$, when the equivalence is immediate.

We illustrate our definition with a sequence of examples (for more detailed exposition and proofs see [KL79] or [CLM76, V]).

Example 4.2.6.

1. By definition, every space is a $L(0)$ space.
2. A space X is $L(1)$, if the canonical map $X \to LX$ admits a retraction μ.
3. Let μ_L denote the structure map $L^2 \to L$. A space X is $L(2)$, if it is $L(1)$ and we have a specified homotopy $I \times L^2 X \to X$ between $\mu\mu_L$ and $\mu(\mu)$. In other words, X is a strictly unital L-algebra in the homotopy category of pointed spaces.
4. A space X is $L(\infty)$ if and only if it is a strong homotopy retract of a L-algebra. If the components of X form a group under the induced multiplication, then X is $L(\infty)$ if and only if it has the homotopy type of an infinite loop space.

There is an obvious analogue of the above definition with based spaces replaced by spectra over S^0 and L replaced by L. Since the category of L-algebras in spectra over S^0 is isomorphic to the cateogory E_∞ ring spectra [May09, 6.2], we obtain an analogous categorical filtration between spectra under S^0 and E_∞ ring spectra.

Applying this equivalence to the definition of $L(n)$ spectra, we see that the definition of an $L(2)$ spectrum is precisely the definition of a strictly unital H_∞ ring spectrum [BMMS86].

The following proposition provides the necessary comparison to prove Theorem 4.1.2.

Proposition 4.2.7.

1. *If X is an $L(2)$ space then $\Sigma^\infty_+ X$ is a strictly unital H_∞ ring spectrum.*

2. *If X is an $L(2)$ space such that the H_∞ ring structure on $\Sigma^\infty_+ X$ rigidifies to an E_∞ structure. Then the $L(2)$ structure on X extends to a $L(\infty)$ structure.*

Proof. The first two parts are obvious from Lemma 4.2.2 and the comments above. For (2), the essential point is the stable maps defining an L structure on $\Sigma^\infty_+ X$ must come from the unstable $L(2)$ structure on X, which implies we are in the image of the functor Σ^∞_+ from L-algebras in spaces to L-algebras in spectra in Lemma 4.2.2. □

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