INTRODUCTION TO ∞-CATEGORIES
EXERCISE SHEET 1

Exercise 1. (a) Show that $\mathrm{h}(\Lambda_i^n) \xrightarrow{\cong} \mathrm{h}(\Delta^n)$ for all $0 < i < n$. Find all pairs (i, n) for which this is false. (b) When is $\mathrm{N}(\mathrm{h}\Lambda_i^n) \to \mathrm{N}(\mathrm{h}\Delta^n)$ a weak equivalence of simplicial sets? (c) Show that an inner anodyne map induces an isomorphism between the homotopy categories.

Exercise 2. Prove that $\mathrm{N}(C)$ is a Kan complex if and only if C is a groupoid.

Exercise 3. (a) Show that the homotopy category of a Kan complex is a groupoid. Describe the category $\mathrm{h}(\mathrm{Sing}(X))$ for a topological space X. (b) Give an example of simplicial set X which is not Kan but $\mathrm{h}(X)$ is a groupoid.

Exercise 4. (a) Prove that (left) homotopy is symmetric. (b) We define the opposite notion of right homotopy as follows: two morphisms $f, g : a \to b$ in an ∞-category X are right homotopic if there is a 2-simplex σ with ordered boundary $\partial\sigma = (g, f, \mathrm{id}_a)$. Prove that two morphisms are left homotopic if and only if they are right homotopic.

Exercise 5. In this exercise we will show that the Yoneda embedding of $\mathcal{I}$ into its category of presheaves exhibits the presheaf category as a free cocompletion of $\mathcal{I}$.

Let $\mathcal{I}$ be a small category and $\mathcal{C}$ be a cocomplete category. Let $\mathcal{P}(\mathcal{I}) := \mathrm{Fun}(\mathcal{I}^{op}, \mathrm{Set})$ be the category of (Set-valued) presheaves on $\mathcal{I}$ and let

$$y\colon \mathcal{I} \to \mathcal{P}(\mathcal{I})$$
$$i \mapsto \hom_{\mathcal{I}}(\cdot, i)$$

denote the Yoneda embedding. Finally, let $\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{I}), \mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{P}(\mathcal{I}), \mathcal{C})$ denote the full subcategory (of a potentially large category) spanned by those functors which preserve all colimits.

Show that the functor $y^*\colon \mathrm{Fun}(\mathcal{P}(\mathcal{I}), \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ restricts to an equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{I}), \mathcal{C}) \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. In particular, $\mathrm{Fun}^{\mathrm{L}}(\mathcal{P}(\mathcal{I}), \mathcal{C})$ is essentially small.

Hint: Show that every presheaf can be canonically exhibited as a colimit of representable presheaves, i.e., presheaves of the form $\mathcal{I}(\cdot, i)$.

Exercise 6. (a) Let $\mathcal{C}$ be a cocomplete category, $F, G : \mathrm{SSet} \to \mathcal{C}$ cocontunuous functors and $\eta : F \to G$ a natural tranformation. Show that η is a natural isomorphism if and only if η_{Δ^n} is a isomorphism for all $n \geq 0$. (b) Deduce that $\mathrm{h}(\mathrm{sk}_2 X) \xrightarrow{\cong} \mathrm{h}(X)$.

Exercise 7. Establish an equivalence between the category of all small simplicial categories and the full subcategory of simplicial objects in categories spanned by objects where the simplicial face and degeneracy maps act by identities on the objects.