INTRODUCTION TO ∞-CATEGORIES
EXERCISE SHEET 2

(1) Regard the poset [1] as a discrete simplicially enriched category and equip it with the symmetric monoidal structure $a \star b = \min(a, b)$ with unit 1.
   (a) Identify the induced Day symmetric monoidal structure $\square$ on $\mathrm{Fun}([1], \mathrm{sSet})$.
   (b) Explicitly identify the commutative monoids in this symmetric monoidal category.
   (c) The Day symmetric monoidal structure is closed. Identify the adjoints to the $\square$ product.
   (d) Fix $\mathcal{D} \subseteq \mathrm{Fun}([1], \mathrm{sSet})$ and $g \in \mathrm{Fun}([1], \mathrm{sSet})$. Let $\mathcal{C} \subseteq \mathrm{Fun}([1], \mathrm{sSet})$ be the class of morphisms f such that $f \square g$ has the left lifting property with respect to $\mathcal{D}$. Show that $\mathcal{C}$ is weakly saturated/cofibrantly closed.
   (e) Take the opposite monoidal structure on [1], i.e., replace max with min above and use 0 as the unit. Identify the induced symmetric monoidal structure on $\mathrm{Fun}([1], \mathrm{sSet})$.

(2) Suppose $i \colon \mathcal{C} \subseteq \mathcal{D}$ is a fully faithful inclusion of small categories. Let $i^* \colon P(\mathcal{D}) \to P(\mathcal{C})$ denote the restriction functor with left adjoint $i_!$ and right adjoint i_*. Show that $i_!$ and i_* are fully faithful.

(3) (a) Show that the class of monomorphisms of simplicial sets is the cofibrant closure/weak saturation of the set of maps $\partial\Delta^n \to \Delta^n$ for all $n \geq 0$. (b) Show that $f \square g$ is a monomorphism if f and g are.

(4) (a) Show that an equivalence of (ordinary) categories induces a Joyal categorical equivalence between the nerves. (b) Prove that a map of ∞-categories is a Joyal categorical equivalence if and only if it is an isomorphism in the naive homotopy category $\mathrm{sSet}^{\mathrm{h}_0}$.

(5) (a) Consider the following notion of weak equivalence between simplicial sets: a map $f : X \to Y$ is a *weak equivalence for Kan complexes* if the induced map

$$f^* : [Y, Z]_{\mathcal{J}} \to [X, Z]_{\mathcal{J}}$$

is bijective for all Kan complexes Z. Show that this is equivalent to the standard notion of weak equivalence of simplicial sets. (b) Conclude that every Joyal categorical equivalence is a weak equivalence.

(6) For $n > 0$ let $I[n] \subseteq \Delta^n$ be the simplicial subset spanned by the edges $(i, i+1)$ for all $0 \leq i < n$. Show that the inclusion $I[n] \to \Delta^n$ is inner anodyne. Conclude that X is an ∞-category if and only if $X^{\Delta^n} \to X^{I[n]}$ is a trivial fibration.

(7) Let $v : C \to D$ be a right anodyne map and $u : A \to B$ a monomorphism. Show that $u \square v$ is right anodyne using the corresponding result for left anodyne maps.