INTRODUCTION TO ∞-CATEGORIES
EXERCISE SHEET 4

(1) Think about why X/b is an ∞-category when X is. Is X/b ever an ∞-category when X isn't? Extend these thoughts to maps $X/b \to X$ for $b : B \to X$. How do ∞-categories make you feel? Express your emotions.

(2) Show that there is a canonical isomorphism

$$h(X_{\mathrm{eq}}) \cong h(X)_{\mathrm{eq}}$$

for any ∞-category X.

(3) The graffiti on the wall reads:

$$"X_{\mathrm{eq}} \star Y_{\mathrm{eq}} \cong (X \star Y)_{\mathrm{eq}}"$$

Prove or disprove. What would you rather write instead? Justify your action.

(4) Another graffiti on the wall reads:

$$"(X/b)_{\mathrm{eq}} \cong X_{\mathrm{eq}}/b"$$

Prove or disprove. Would you rather write something else?

(5) Let $p : E \to S$ be a left fibration. Suppose that for every morphism $u : s \to s'$ in S, the map

$$u_! : E_s \to E_{s'}$$

is an isomorphism in the homotopy category of Kan complexes. The purpose of the Exercise is to show that p is a Kan fibration.

(a) Show that it suffices to prove that

$$q : E^{\Delta^1} \to E^{\{1\}} \times_{S^{\{1\}}} S^{\Delta^1}$$

is a trivial fibration.

(b) Conclude that it suffices to prove that q has contractible fibers.

(c) Let $u : s \to s'$ be a morphism in S and X be the simplicial set of sections of the projection

$$E \times_S \Delta^1 \to \Delta^1$$

where the fiber product is defined by $u : \Delta^1 \to S$. There is an obvious map $q' : X \to E_{s'}$ defined by evaluation at $\{1\}$. Show that q and q' have the same fiber over points (e, u) and $e \in E_{s'}$ respectively.

(d) Show that q' is a Kan fibration.

(e) Use the assumptions to conclude that q' is a homotopy equivalence and finish the proof.

(6) Bonus question: Let

$$G = \mathfrak{C}[x \star \Delta^n](x, -)\colon \mathfrak{C}[x \star \Delta^n] \to \mathrm{sSet}$$

and let $F\colon \mathfrak{C}[\Delta^n] \to \mathrm{sSet}$ be the simplicial functor obtained by pulling back along the inclusion $d^{n+1}\colon \Delta^n \to x \star \Delta^n$.

(a) Show the colimit of F calculated as a sSet enriched functor is $\mathrm{Str}_*\Delta^n$; the straightening functor from the covariant model structure on $\mathrm{sSet} = \mathrm{sSet}_{/*}$ to sSet.

(b) Explicitly calculate this colimit and identify it as Δ^n.

(c) Show that although the functor $\mathrm{Str}_*\Delta^{(-)}\colon \Delta \to \mathrm{sSet}$ is object-wise the standard inclusion, this inclusion does not respect the degeneracy maps.