INTRODUCTION TO ∞-CATEGORIES
EXERCISE SHEET 5

(1) (a) Let $f, g : X \to Y$ be maps between ∞-categories. Show that if they are Joyal homotopic (i.e. homotopic with respect to $\mathcal{J}$) then the maps $f_{\mathrm{eq}}, g_{\mathrm{eq}} : X_{\mathrm{eq}} \to Y_{\mathrm{eq}}$ are also Joyal homotopic. Conclude that a Joyal equivalence between ∞-categories induces a homotopy equivalence between the respective maximal Kan subcomplexes. (b) Show that a homotopy equivalence between Kan complexes is a Joyal categorical equivalence.

(2) Consider the following generalization of the Special Outer Horn Lifting Theorem ("Main Theorem"). Let $p : X \to Y$ be an inner fibration between ∞-categories and suppose we have a lifting diagram

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

where $u_{|\{0,1\}}$ is an equivalence in X. (a) Define the simplicial set of lifts for this diagram. (b) Prove or disprove: the simplicial set of lifts is contractible. (Hint: Try first the case where $Y = \Delta^0$.)

(3) Let $p : X \to Y$ be a categorical fibration between ∞-categories and suppose that p detects equivalences (i.e. $p(f)$ equivalence $\Rightarrow f$ equivalence).
   (a) Show that if Y is a Kan complex, then p is a Kan fibration.
   (b) Show that the fibers of p are Kan complexes.
   (c) Compare with the corresponding results for left (or right) fibrations.

(4) Let $p : X \to Y$ be a Joyal fibration between *simplicial sets*, i.e. a map which has the right lifting property with respect to every map which is a monomorphism and a categorical equivalence, and $u : A \to B$ a monomorphism. Prove that

$$\langle u, p\rangle : X^B \to X^A \times_{Y^A} Y^B$$

is a categorical fibration and that it is a trivial fibration if u is categorical equivalence. (Hint: Note that this was shown in the lectures in the case where X and Y are ∞-categories. For the general case, use the characterization for the class of maps which are both monorphisms and categorical equivalences which was discussed in the lectures.)

(5) Let P be the subposet of $[m] \times [n]$ which contains all the elements except $(0, n)$. The following assertion was used in the lectures: the inclusion map

$$(\Lambda^m_m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \to (\Lambda^m_m \times \Delta^n) \cup (\Delta^m \times \partial\Delta^n) \cup N(P)$$

where all of the objects are regarded as subobjects of $\Delta^m \times \Delta^n$, is inner anodyne for $n > 0$. A proof is in Joyal's Barcelona Notes (Proposition H.0.21). Read the proof and produce an independent exposition minimizing the amount of prerequisites.

(6) In this exercise, you are required to supply complete proofs of some results about locally presentable categories that were presented in the letures.

(a) Let $\mathcal{C}$ be a locally λ-presentable category and $\mathcal{A}$ the (essentially) small full subcategory spanned by the λ-presentable objects. The inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ extends uniquely to an adjunction

$$F : \mathrm{Set}^{\mathcal{A}^{\mathrm{op}}} \leftrightarrows \mathcal{C} : G.$$

(i) Explain the definitions of the functors F and G.

(ii) Show that G preserves λ-directed colimits.

(iii) Show that G is full and faithful. (Hint: Identify the components of the counit transformation with the map from the canonical colimit of $\mathcal{A}$-objects.)

Conclude that locally presentable categories are complete.

(b) Let $\mathcal{C}$ be a category which admits λ-directed colimits and $\mathcal{A}$ a small category. Suppose that

$$F : \mathrm{Set}^{\mathcal{A}^{\mathrm{op}}} \leftrightarrows \mathcal{C} : G$$

is an adjunction such that G preserves λ-directed colimits and G is full and faithful. Prove that $\mathcal{C}$ is locally λ-presentable.

(c) Use (b) to prove that Grothendieck topoi are locally presentable categories. (Hint: Characterize sheaves by a right lifting or orthogonality property with respect to the covering sieves in order to show that the category of sheaves is closed in the category of presheaves under large enough directed colimits.)

(7) The purpose of this exercise is to construct an alternative "path space factorization".

(a) Let X be an ∞-category.

(i) Prove that the map

$$(\mathrm{ev}_0, \mathrm{ev}_1) : X^{\mathcal{J}} \to X \times X,$$

given by evaluation at the 0-simplices of $\mathcal{J}$, is a categorical fibration.

(ii) Prove that ev_0 and ev_1 are trivial fibrations.

(b) Let $f : X \to Y$ be a map between ∞-categories. Define $Q(f)$ by the following pullback

$$\begin{array}{ccc} Q(f) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ Y^{\mathcal{J}} & \xrightarrow{\mathrm{ev}_0} & Y \end{array}$$

Show that there is a natural factorization of f as

$$X \xrightarrow{j_f} Q(f) \xrightarrow{p_f} Y$$

where j_f is a categorical equivalence and p_f is a categorical fibration.