Equivariant cohomology of representation spheres and $\text{Pic}(S_G)$-graded homotopy groups

Justin Noel

Max Planck Institute for Mathematics

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Question

What should homology and cohomology groups be indexed over?

Normally $H^* (X)$ and $H_* (X)$ is graded over $\mathbb{N}$. Generalized (co)homology (e.g., $K$-theory) is graded over $\mathbb{Z}$. Equivariant (co)homology is graded over $\mathbb{N}$ and sometimes $RO(G)$. 

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UniBonn, MPIM
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Justin Noel  
UniBonn, MPIM
One way to think of gradings

- Cohomology theories ↔ spectra.
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- Cohomology theories $\leftrightarrow$ spectra.
- Suppose $X \in \text{Top}_* \subset \text{Spectra}$, $M \in \text{AbGroup}$.

Similarly for generalized (co)homology.
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UniBonn, MPIM
Indexing by spheres

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$$\tau : S^i \wedge S^j \rightarrow S^j \wedge S^i$$

which has degree $(-1)^{i,j}$.
Why spheres?

- They are small, so wedge axiom holds:

\[ E_i(\bigvee X_j) \cong \bigoplus_{j \in J} E_i(X_j). \]
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\[ \cong [S^{i-1}, E \wedge X] \]
\[ \cong E_{i-1}(X). \]
Requirements for indices

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2. Suspension axiom $\implies$ Indexing objects are invertible.
3. The (abelian) group of such objects is called the Picard group.
Päuschen
Can define Picard group, \( \text{Pic}(\mathcal{C}) \) for any symmetric monoidal category \( \mathcal{C} \) (dualizable, w/ inverses).
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Can define Picard group, $\text{Pic}(\mathcal{C})$ for any symmetric monoidal category $\mathcal{C}$ (dualizable, w/ inverses).

$\text{Pic}(S) \cong \mathbb{Z}$, so nothing new here.

Want ‘more spheres.’
Representation spheres

Let $G$ be a finite group.
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Take an orthogonal $G$ representation $V = \mathbb{R}^n \odot G$. 

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![Diagram of a unit disc with colored sections representing a non-trivial representation of $C_5$.]
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![Unit Disc Representation](image-url)
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E.g., the color slices above.
Construction gives a morphism $RO(G) \to \text{Pic}(S_G)$. 
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Factors as

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$$RO(G) \xrightarrow{\sim} JO(G) \xhookrightarrow{} \text{Pic}(S_G).$$

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Let us find a toy case where we can compute groups indexed over Pic($S_G$).
Known results

Theorem (tom Dieck-Petrie)

\[ \text{Rank } JO(G) = \text{Rank } \text{Pic}(S_G) \iff G \text{ is nilpotent.} \]
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**Theorem (tom Dieck-Petrie)**

\[ \text{Rank } JO(G) = \text{Rank Pic}(S_G) \iff G \text{ is nilpotent.} \]

**Theorem (Kawakubo)**

\[ JO(G) \cong \text{Pic}(S_G) \iff G = C_n \text{ or } D_{2 \cdot 2^n}. \]
Theorem (Kawakubo)

\[ \text{JO}(S_{C_n}) \cong \text{Pic}(S_{C_n}) \cong \bigoplus_{d \mid n} \left( \mathbb{Z} \oplus \left( \mathbb{Z}/d\mathbb{Z}^{\times} / \langle \pm 1 \rangle \right) \right). \]
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- Torsion free summands generated by a rotation of order \(d\).
\[ \text{Theorem (Kawakubo)} \]

\[ JO(S_{C_n}) \cong \text{Pic}(S_{C_n}) \cong \bigoplus_{d|n} \left( \mathbb{Z} \oplus (\mathbb{Z}/d\mathbb{Z}^\times /\langle \pm 1 \rangle) \right). \]

- Torsion free summands generated by a rotation of order \( d \).
- Torsion summand generated by differences of such representations.
Let’s compute the Pic($S_G$)-graded homotopy of something.
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How do we do this?
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How do we do this?

What should $M$ be equivariantly?
Given a CW-complex $X$ let

$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\},$$
Reminder: Cellular homology

- Given a CW-complex $X$ let
  
  $$\tilde{C}_i(X) := \mathbb{Z}\{i\text{-cells of } X\},$$
  
  $$\tilde{C}_0(X) := \ker(C_0(X) \to C_0(*)).$$
Reminder: Cellular homology

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- $H_\ast(X) \cong H_\ast \left[ \cdots C_{i+1}(X) \overset{\partial}{\to} C_i(X) \overset{\partial}{\to} C_{i-1}(X) \cdots \overset{\partial}{\to} \tilde{C}_0(X) \right].$
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- $H_\ast(X) \cong H_\ast \left[ \cdots \to C_{i+1}(X) \xrightarrow{\partial} C_i(X) \xrightarrow{\partial} C_{i-1}(X) \to \cdots \xrightarrow{\partial} \tilde{C}_0(X) \right]$.

- $H^\ast(X)$ is calculated by taking the dual of this complex and then taking cohomology.
Bredon homology with coefficients in \( \mathbb{Z} \)

Given a \( G \)-CW-complex \( X \) let

\[ C_i(X) := \mathbb{Z}\{i\text{-cells of } X\} \]
Given a $G$-CW-complex $X$ let

$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\} \circ G.$$
Bredon homology with coefficients in $\mathbb{Z}$

- Given a $G$-CW-complex $X$ let

$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\} \otimes G.$$  

- For a subgroup $K \leq G$, let $C_i^K(X) \subset C_i(X)$ be the subgroup of $K$-invariant chains.
Bredon homology with coefficients in $\mathbb{Z}$

- Given a $G$-CW-complex $X$ let

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$$H^K_*(X) \cong H_* \left[ \cdots \frac{C^{K}_{i+1}(X)}{\partial} \xrightarrow{\partial} C^K_i(X) \xrightarrow{\partial} C^K_{i-1}(X) \cdots \xrightarrow{\partial} \tilde{C}^K_0(X) \right].$$
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- When $K$ is the trivial subgroup: $H^K_*(X) \cong H_*(X)$. 

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Bredon homology with coefficients in $\mathbb{Z}$

- Given a $G$-CW-complex $X$ let

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- For cohomology first take the invariants on the cochains.
Mackey functors

- Other coefficient systems?
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Mackey functors

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- Want \( H_i^{(-)}(X) \) to also be an acceptable coefficient system.
- These functors assign an abelian group to each subgroup of \( G \) and have induction, restriction, and action maps, satisfying some axioms.
Mackey functors

- Other coefficient systems?
- Want $H_i^{(-)}(X)$ to also be an acceptable coefficient system.
- These functors assign an abelian group to each subgroup of $G$ and have induction, restriction, and action maps, satisfying some axioms.
- Such functors should form an abelian category.
Definition: Mackey functors

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$\mathcal{O}(G)$ is the category of finite $G$-sets and $G$-morphisms.
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A Mackey functor is a pair

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M^* : \mathcal{O}(G)^\text{op} \to \text{AbGroup}
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- \( M^*(X \sqcup Y) = M^*(X) \times M^*(Y) \).
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- \( M \) satisfies a double coset formula.
Definition explained

- Alternatively, $M$ assigns to each $H \leq G$ an abelian group $M(G/H)$.
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- $M(G/H)$ has action of $W_{GH} = N_{GH}/H$. 

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- From $H \leq K$ we obtain a map $\pi: G/H \to G/K$ inducing a restriction maps

\[ M(G/H) \xrightarrow{M_\ast \pi = \text{Res}^K_H} M(G/K) \]
Definition explained

- Alternatively, $M$ assigns to each $H \leq G$ an abelian group $M(G/H)$.
- $M(G/H)$ has action of $W_G H = N_G H / H$.
- From $H \leq K$, we obtain a map $\pi : G/H \to G/K$ inducing a restriction maps

$$M(G/H) \xrightarrow{M_* \pi = \text{Res}^K_H} M(G/K)$$

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- From $H \leq K$ we obtain a map $\pi : G/H \to G/K$ inducing a restriction maps

$$M(G/H) \xrightarrow{M_* \pi = \text{Res}^K_H} M(G/K)$$

and a transfer map

$$M(G/K) \xrightarrow{M_* \pi = \text{Ind}^K_H} M(G/H).$$
Example: $G$-Modules

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Example: $G$-Modules

- Given a $G$-module $M$ we can construct a functor $M(-)$:
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  $M(G/H) = \text{Mod}_G(\mathbb{Z}[G/H], M) \cong M^H$.

- For $K \leq H$,
  
  $$M(G/H) = M^H \xrightarrow{\text{Res}^H_K} M(G/K) = M^K$$

  induced by quotient map

  $$q : \mathbb{Z}[G/K] \to \mathbb{Z}[G/H].$$
Example: $G$-Modules

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  $$M(G/H) = \text{Mod}_G(\mathbb{Z}[G/H], M) \cong M^H.$$
- For $K \leq H$,
  $$M(G/H) = M^H \xrightarrow{\text{Res}_K^H} M(G/K) = M^K$$
  induced by quotient map
  $$q : \mathbb{Z}[G/K] \to \mathbb{Z}[G/H].$$
- We also have a transfer
  $$M(G/K) = M^K \xrightarrow{\text{Ind}_K^H} M(G/H) = M^H$$
Example: $G$-Modules

- Given a $G$-module $M$ we can construct a functor $M(-)$:
  - $M(G/H) = \mathcal{M}od_G(\mathbb{Z}[G/H], M) \cong M^H$.
- For $K \leq H$,
  \[
  M(G/H) = M^H \xrightarrow{\text{Res}_K^H} M(G/K) = M^K
  \]
  induced by quotient map
  \[
  q : \mathbb{Z}[G/K] \to \mathbb{Z}[G/H].
  \]
- We also have a transfer
  \[
  M(G/K) = M^K \xrightarrow{\text{Ind}_K^H} M(G/H) = M^H
  \]
  induced by summing over the fibers of $q$. 
Example: Burnside ring

Definition

Let $A(G)$ be the Grothendieck group of finite $G$-sets up to iso.
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- Crossing with $G/H$ defines $\text{Ind}^G_H$. 
Note

$\pi^*HA$

$A(C_9)$

$X = \bigoplus_i G/H_i$ (Decomposition into orbits).

Rank $A(G) = \# \text{Conjugacy classes of subgroups of } G$.

Example (Multiplication Table)

$A(C_9) = \mathbb{Z}\{C_9/C_9, C_9/C_3, C_9\}$

$C_9 \times C_9$ is a free $C_9$ set with 81 elements.

Justin Noel

UniBonn, MPIM
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- $A(H)$ is a ring such that $A$ is a commutative Green functor.
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- Analogue of $\mathbb{Z}$ equivariantly.
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**Definition**

Given a Mackey functor $M$,
Induced Mackey functors

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Given a Mackey functor $M$ let $M \otimes G/H$ be the Mackey functor defined by

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Example

$A \otimes G \cong \mathbb{Z}[G]$
Given a $G$-CW-complex $X$ let

$$C_i(X) : M \otimes \{i\text{-cells of } X\} \circlearrowleft G.$$
Bredon homology with coefficients in $M$

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$$H^K_*(X) \cong H_* \left[ \cdots C_{i+1}(X)(G/K) \xrightarrow{\partial} C_i(X)(G/K) \xrightarrow{\partial} C_{i-1}(X)(G/K) \cdots \xrightarrow{\partial} \tilde{C}_0(X)(G/K) \right]$$
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  \]

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- For cohomology one takes a dual complex, with $\text{Ind}^G_H \leftrightarrow \text{Res}^G_H$. 
Päuschen
Fix a finite group $G$ and determine explicit models for all of the irreducible real representations of $G$. 
Method of computating $\pi_\star HA$

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Assemble the computations to compute $H^*_G(S^V+W)$. 
Consider $\rho_n$ the rotation of order $n$ ($n > 2$)
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\[
\sum \quad \Rightarrow
\]

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\[
\begin{array}{c}
A \xleftarrow{\varepsilon} A \otimes C_n \xrightarrow{g^{-1}} A \otimes C_n \\
\sum
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\[ H^e_*(S^{\rho_n}; A) \]

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Evaluate at $C_n/e$: 

$H^e_0(S^{\rho_n}) \cong 0$ 

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$H^e_2(S^{\rho_n}) \cong Z$
\[ H_e^*(S^{\rho_n}; A) \]

\[
A \leftarrow A \otimes C_n \xrightarrow{g^{-1}} A \otimes C_n
\]

Evaluate at \( C_n/e \):

\[
\mathbb{Z} \quad \mathbb{Z}|C_n| \quad \mathbb{Z}|C_n|
\]
\[ H^e_\ast(S^{\rho_n}; A) \]

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Evaluate at \( C_n/e \):

\[ \mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}|C_n| \xrightarrow{[g]-[1]} \mathbb{Z}|C_n| \]
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Get $H_*(S^2)$ as expected.
$H_{\ast}^{C_n}(S^{\rho_n}; A)$

$A \xleftarrow{\epsilon} A \otimes C_n \xleftarrow{g^{-1}} A \otimes C_n$
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\[ A \xleftarrow{\epsilon} A \otimes C_n \xrightarrow{g^{-1}} A \otimes C_n \]

Evaluate at $C_n/C_n$: 
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Evaluate at \( C_n/C_n \):

\[ A(C_n) \quad \mathbb{Z} \quad \mathbb{Z} \]
\[ H^*_{\ast}(S^{\rho_n};A) \]

\[ A \leftarrow A \otimes C_n \xrightarrow{g^{-1}} A \otimes C_n \]

Evaluate at \( C_n/C_n \):

\[ A(C_n) \xleftarrow{\text{Ind}_{e}^{C_n}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \]
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\[ A(C_n) \xleftarrow{\text{Ind}^C_n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \]

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\( \pi_1 \mathcal{H} \)

\[ H^*_C(S^{\rho n}; A) \]

\[ A \xrightarrow{\Delta} A \otimes C_n \xrightarrow{g^{-1} - 1} A \otimes C_n \]
Evaluate at $C_n/C_n$: 

$$A \xrightarrow{\Delta} A \otimes C_n \xrightarrow{g^{-1} - 1} A \otimes C_n$$
\[ H^*_{C_n} (S^\rho_n ; A) \]

\[ A \xrightarrow{\Delta} A \otimes C_n \xrightarrow{g^{-1} - 1} A \otimes C_n \]

Evaluate at \( C_n / C_n \):

\[ A(C_n) \xrightarrow{\sim} \mathbb{Z} \quad \mathbb{Z} \]

\[ \pi_* \mathcal{H} \]
\[ H^*_{C_n}(S^{\rho_n}; A) \]

\[ A \xrightarrow{\Lambda} A \otimes C_n \xrightarrow{g^{-1}-1} A \otimes C_n \]

Evaluate at \( C_n/C_n \):

\[ A(C_n) \xrightarrow{\text{Res}^c_{C_n}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \]
\[ H^*_{C_n} (S^\rho_n ; A) \]

\[
A \xrightarrow{\Delta} A \otimes C_n \xrightarrow{g^{-1} - 1} A \otimes C_n
\]

Evaluate at \( C_n/C_n \):

\[
A(C_n) \xrightarrow{\text{Res}_{C_n}} \mathbb{Z} \to \mathbb{Z}
\]

\[ H^0_{C_n} (S^\rho_n ) \cong \tilde{A}(C_n) = \ker(A(C_n) \xrightarrow{\text{Res}_{C_n}} \mathbb{Z}) \]
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\[ H^2_C(S^{\rho_n}) \cong \mathbb{Z}. \]
Assembling the computation

Once we know $C_\ast(S^V)$ and $C_\ast(S^W)$ have
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Once we know $C_*(S^V)$ and $C_*(S^W)$ have

$$H_*(S^V + W) \cong H_*(C_*(S^W) \otimes C_*(S^V))$$
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$$E_{s,t}^2 = H_s(S^V; H_t(S^W)) \implies H_{s+t}(S^{V+W}),$$
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...to simplify $E_1$. 
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3 (Functoriality) Use subgroup functoriality to determine the differentials and multiplicative relations.
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1. (Reverse induction) If $V$ is the pullback of a representation of $G/H$, then can pullback complex.

2. (Reciprocity) Use the formula

$$S^W \wedge \text{Ind}_H^G S^i \cong \text{Ind}_H^G \left( \text{Res}_H^G (S^W) \wedge S^i \right)$$

3. Use subgroup functoriality to determine the differentials and multiplicative relations.

4. (Competing computations) Decompose the representation in different ways.
$H_*^{C_n}(S^{2\rho_n})$

$M \xleftarrow{\epsilon} M \otimes C_n \xrightarrow{g^{-1}} M \otimes C_n$
\[ H^C_n(S^{2\rho_n}) \]

\[
M \overset{\epsilon}{\leftarrow} M \otimes C_n \overset{g^{-1}}{\rightarrow} M \otimes C_n
\]

Evaluate at \( C_n/C_n \) with \( M = H_*(S^{\rho_n}) \):
\[ M \xleftarrow{\epsilon} M \otimes C_n \xrightarrow{g^{-1}} M \otimes C_n \]

Evaluate at \( C_n/C_n \) with \( M = H_\ast(S^{\rho_n}) \):

- \( H^C_\ast(S^{\rho_n}) \)
- \( H^e_\ast(S^2) \)
- \( H^e_\ast(S^2) \)
$H^C_{n}(S^{2\rho_n})$

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Evaluate at $C_n/C_n$ with $M = H_*(S^{\rho_n})$:

\[ H^C_{n}(S^{\rho_n}) \xleftarrow{\text{Ind}_{C_n}^{C_n}} H^e_*(S^2) \xrightarrow{0} H^e_*(S^2) \]
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H_{\star}^C(S^{\rho_n}) \xleftarrow{\text{Ind}_{e}^{C_n}} H_{\star}^e(S^2) \xrightarrow{0} H_{\star}^e(S^2)
\]

\[
H_{0}^C(S^{2\rho_n}) \cong A(C_n)/[C_n]
\]
\[ H^C_{\ast}(S^{2\rho_n}) \]

\[ M \xleftarrow{\epsilon} M \otimes C_n \xrightarrow{g^{-1}} M \otimes C_n \]

Evaluate at \( C_n/C_n \) with \( M = H_{\ast}(S^{\rho_n}) \):

\[ H^C_{\ast}(S^{\rho_n}) \xleftarrow{\text{Ind}_{C_n}^e} H^e_{\ast}(S^2) \xrightarrow{0} H^e_{\ast}(S^2) \]

\[ H^C_{0}(S^{2\rho_n}) \cong A(C_n)/[C_n] \]
\[ H^C_{\text{odd}}(S^{2\rho_n}) \cong 0 \]
\[ \pi_* HA \]

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\[ H^C_n(S^{2\rho_n}) \cong 0 \]

\[ H^C_n(S^{2\rho_n}) \cong \mathbb{Z}/(n) \]
\[ H_{\ast}^{C_n}(S^{2\rho_n}) \]

\[ M \leftrightarrow M \otimes C_n \xrightarrow{g^{-1}} M \otimes C_n \]

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\[ H_{\text{odd}}^{C_n}(S^{2\rho_n}) \cong 0 \]

\[ H_{2}^{C_n}(S^{2\rho_n}) \cong \mathbb{Z}/(n) \]

\[ H_{4}^{C_n}(S^{2\rho_n}) \cong \mathbb{Z} \]
One can compute these groups inductively and get a regular pattern.
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$H_0^{C_n}(S^{m \rho_n}; A) \cong A(C_n)/([C_n])$. 

$H_{odd}^{C_n}(S^{m \rho_n}) \cong 0$. 

Can also compute directly from a single chain complex.
One can compute these groups inductively and get a regular pattern.

- $H^C_n(S^m \rho_n; A) \cong A(C_n)/[C_n]$.
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Päuschen
There are also external products

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These relate the (co)homology groups of different representation spheres.
Multiplicative relations

- There are also external products
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- Can deduce from spectral sequence and Green functor relations.
Some Relations

\[ i_*(1) = a_V \in H_0^G(S^V) \text{ induced by inclusion of } S^0 \to S^V. \]
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Computations for $G = C_2, \ M = \mathbb{Z}$
Computations for $G = C_2$, $M = \mathbb{Z}$
Computations for $G = C_2$, $M = \mathbb{Z}$
Computations for $G = C_2 \ M = \mathbb{Z}$
Brace yourselves
\[ H^C_{\ast} (S^{\rho_5 - \rho_5 \otimes_k}; A) \]

\[ C_{\ast}(S^{\rho_5}) \otimes A \]

\[
\begin{array}{c c c c}
0 & 1 & 2 \\
\end{array}
\]

\[ A \xleftarrow{\epsilon} A \otimes C_5 \xleftarrow{g^{-1}} A \otimes C_5 \]
\[ H^C_5(S^{\rho_5 - \rho_5 \otimes k}; A) \]

\[ C^*(S^{\rho_5}) \otimes A \cong C_{-\ast}(S^{-\rho_5}) \otimes A \]

\[
\begin{array}{ccc}
0 & & \\
0 & A & \\
& \Delta & \\
-1 & A \otimes C_5 & \\
& g^k - 1 & \\
-2 & A \otimes C_5 & \\
\end{array}
\]
\( H_{*}^{C_5}(S^{\rho_5 - \rho_5 \otimes k}; A) \)

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & A & A \otimes C_5 & A \otimes C_5 \\
1 & A \otimes C_5 & A \otimes C_5 \otimes C_5 & A \otimes C_5 \otimes C_5 \\
2 & A \otimes C_5 & A \otimes C_5 \otimes C_5 & A \otimes C_5 \otimes C_5 \\
\end{array}
\]

Diagonal gives total degree.
$H^C_5(\mathcal{S}^{\rho_5 - \rho_5^k}; A)$

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$H^C_{\ast}(S^{\rho_5-\rho^\otimes_k_5};A)$

Compute cohomological (↓) direction first.
Compute cohomological (\(\downarrow\)) direction first.

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<th></th>
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<tr>
<td>0</td>
<td>(\tilde{A}(C_5))</td>
<td>0</td>
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<tr>
<td>(-1)</td>
<td>0</td>
<td>0</td>
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<td>(-2)</td>
<td>(\mathbb{Z})</td>
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</table>
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\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & ? & \mathbb{Z} & ? & \mathbb{Z}
\end{array}
\]
Compute homological (←) direction first.
\[ \pi_* \mathcal{H} A \]

\[ H^C_{C^5}(S^{\rho_5 - \rho_5^k}; A) \]

Compute homological (←) direction first.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & A(C_5)/[C_5] & 0 & \mathbb{Z} \\
-1 & 0 & 0 & \mathbb{Z} \\
-2 & 0 & 0 & \mathbb{Z}
\end{array}
\]
Compute homological (←) direction first.

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\begin{array}{ccc}
0 & 1 & 2 \\
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-1 & 0 & 0 & \mathbb{Z} \\
-2 & 0 & 0 & \mathbb{Z}
\end{array}
\]
Resolve differentials for first spectral sequence.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & A(C_5)/[C_5] & 0 \\
-1 & 0 & 0 \\
-2 & 0 & 0 \\
\end{array}
\]

\[\mathbb{Z}\]

\[\mathbb{Z}\]

\[\mathbb{Z}\]

\[\mathbb{Z}\]
Resolve differentials for second spectral sequence.
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\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & Z & Z & Z \\
\end{array}
\]

\[
\begin{array}{ccc}
& 1 \\
Z & \leftarrow & Z \\
& 0 \\
\end{array}
\]
Resolve differentials for second spectral sequence.

\[ \begin{array}{ccc}
0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & 0 & 0 & \mathbb{Z}
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Resolve differentials for second spectral sequence.

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<tr>
<td>0</td>
<td>Ā((C_5))</td>
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<td>0</td>
<td>(\mathbb{Z})</td>
</tr>
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</table>

We have an extension:

\[
0 \rightarrow \tilde{A}(C_5) \rightarrow H_0^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A) \rightarrow \mathbb{Z} \rightarrow 0
\]
Our extension problem:

$$0 \to \tilde{A}(C_5) \to H^C_{\ast}(S^5 - \rho_5^k ; A) \to \mathbb{Z} \to 0$$
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\[ 0 \to \tilde{A}(C_5) \to H^C_0(S^{\rho_5 - \rho_5^k}) \to \mathbb{Z} \to 0 \]

Extension splits additively, but not as \( A(C_5) \) modules.
Our extension problem:

\[ 0 \to \tilde{A}(C_5) \to H^C_{0}(S^{\rho_5 - \rho_5^k}) \to \mathbb{Z} \to 0 \]

- Extension splits additively, but not as \( A(C_5) \) modules.
- Use bicomplex to solve extension.
\( H^C_5(S^0, \rho^5_5 \otimes_k S^0) \)

- Our extension problem:

\[
0 \rightarrow \tilde{A}(C_5) \rightarrow H^C_5(S^0 - \rho^5_5 \otimes_k S^0) \rightarrow \mathbb{Z} \rightarrow 0
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- If \(k = \pm 1 \mod 5\) get \(A(C_5) = H^C_5(S^0)\).
Our extension problem:

\[ 0 \to \tilde{A}(C_5) \to H^C_{C_5}(S^{\rho_5 - \rho^\otimes_k}) \to \mathbb{Z} \to 0 \]

- Extension splits additively, but not as \( A(C_5) \) modules.
- Use bicomplex to solve extension.
- If \( k = \pm 1 \mod 5 \) get \( A(C_5) = H^C_0(S^0) \).
- If \( k = \pm 2 \mod 5 \) get a projective \( A(C_5) \) module of rank one.
We can also determine explicit models for the irreducible real representations of $C_n$, $D_n$, $A_4$, and $S_4$. 
Summary

- We can also determine explicit models for the irreducible real representations of $C_n$, $D_n$, $A_4$, and $S_4$.
- We can compute the homology and cohomology of these representation spheres.
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- Calculating $H^*_G(S^V+W)$ is generally difficult due to complications in the spectral sequence.
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We can compute the homology and cohomology of these representation spheres.

Calculating $H^G_\ast(S^V+W)$ is generally difficult due to complications in the spectral sequence.

There is still plenty of other computations left to do.
Twisted tetrahedral representation of $\Sigma_4$
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Twisted tetrahedral representation of $\Sigma_4$