Hopf Algebras and the Yoneda Lemma

Justin Noel

Department of Mathematics
University of Chicago

3/30/2008
Graduate Student Topology Conference

Our Goals

- Compute the cohomology of some classifing spaces of vector bundles.
- Compute the maps on cohomology induced by $\oplus$ and $\otimes$.
- Actually, we will look at the functors $\mathrm{Ring}(E^*(X), -)$ and compute the maps there.
- Identify these functors.

Our Goals

- Compute the cohomology of some classifing spaces of vector bundles.
- Compute the maps on cohomology induced by $\oplus$ and $\otimes$.
- Actually, we will look at the functors $\mathrm{Ring}(E^*(X), -)$ and compute the maps there.
- Identify these functors.

Our Goals

- Compute the cohomology of some classifing spaces of vector bundles.
- Compute the maps on cohomology induced by $\oplus$ and $\otimes$.
- Actually, we will look at the functors $\mathrm{Ring}(E^*(X), -)$ and compute the maps there.
- Identify these functors.

Our Goals

- Compute the cohomology of some classifing spaces of vector bundles.
- Compute the maps on cohomology induced by $\oplus$ and $\otimes$.
- Actually, we will look at the functors $\mathrm{Ring}(E^*(X), -)$ and compute the maps there.
- Identify these functors.

Outline

- Vector bundles and the classifying spaces $BU(n)$.
- Review generalized cohomology theories.
- The Yoneda embedding.
- Identifying our functors.
- Tying it all together.

Outline

- Vector bundles and the classifying spaces $BU(n)$.
- Review generalized cohomology theories.
- The Yoneda embedding.
- Identifying our functors.
- Tying it all together.

Outline

- Vector bundles and the classifying spaces $BU(n)$.
- Review generalized cohomology theories.
- The Yoneda embedding.
- Identifying our functors.
- Tying it all together.

Outline

- Vector bundles and the classifying spaces $BU(n)$.
- Review generalized cohomology theories.
- The Yoneda embedding.
- Identifying our functors.
- Tying it all together.

Outline

- Vector bundles and the classifying spaces $BU(n)$.
- Review generalized cohomology theories.
- The Yoneda embedding.
- Identifying our functors.
- Tying it all together.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \mathrm{colim}\, BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \mathrm{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \mathrm{colim}\, BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \mathrm{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \operatorname{colim} BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \operatorname{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \operatorname{colim} BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \operatorname{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \operatorname{colim} BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \operatorname{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

The Classifying Spaces $BU(n)$ and BU

- Assume X is a based, finite dimensional CW-complex.
- A complex vector bundle of rank n over X corresponds to an element of $[X, BU(n)]$.
- The operations $\oplus$ and $\otimes$ on vector bundles give rise to maps $\oplus : BU(n) \times BU(m) \to BU(n+m)$ and $\otimes : BU(m) \times BU(n) \to BU(nm)$.
- By adjoining a rank one trivial bundle we obtain a map $BU(n) \to BU(n+1)$.
- The space $BU = \mathrm{colim}\, BU(n)$, represents stable equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$.
- The space $BU \times \mathbb{Z} = \mathrm{colim} \coprod BU(n)$, represents formal differences of equivalence classes of vector bundles and inherits the operations $\oplus$ and $\otimes$, which make it a ring up to coherent homotopy.

Chern Classes

- Some calculations:
  - $H^*(BU(1)) \cong \mathbb{Z}[x]$.
  - $H^*(BU(n)) \cong H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[c_1, \ldots, c_n]$.
- Given a vector bundle η over X, we have a classifying map $X \to BU(n)$, which gives us a map $H^*(BU(n)) \to H^*(X)$.
- Such a map is determined by $c_1(\eta), \ldots, c_n(\eta) \in H^*(X)$. These are the Chern classes of η.
- The $c_i(\eta)$ are obstructions to the trivializability of η.
- We want to generalize this framework to other cohomology theories.

Chern Classes

- Some calculations:
  - $H^*(BU(1)) \cong \mathbb{Z}[x]$.
  - $H^*(BU(n)) \cong H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[c_1, \ldots, c_n]$.
- Given a vector bundle η over X, we have a classifying map $X \to BU(n)$, which gives us a map $H^*(BU(n)) \to H^*(X)$.
- Such a map is determined by $c_1(\eta), \ldots, c_n(\eta) \in H^*(X)$. These are the Chern classes of η.
- The $c_i(\eta)$ are obstructions to the trivializability of η.
- We want to generalize this framework to other cohomology theories.

Chern Classes

- Some calculations:
  - $H^*(BU(1)) \cong \mathbb{Z}[x]$.
  - $H^*(BU(n)) \cong H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[c_1, \ldots, c_n]$.
- Given a vector bundle η over X, we have a classifying map $X \to BU(n)$, which gives us a map $H^*(BU(n)) \to H^*(X)$.
- Such a map is determined by $c_1(\eta), \ldots, c_n(\eta) \in H^*(X)$. These are the Chern classes of η.
- The $c_i(\eta)$ are obstructions to the trivializability of η.
- We want to generalize this framework to other cohomology theories.

Chern Classes

- Some calculations:
  - $H^*(BU(1)) \cong \mathbb{Z}[x]$.
  - $H^*(BU(n)) \cong H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[c_1, \ldots, c_n]$.
- Given a vector bundle η over X, we have a classifying map $X \to BU(n)$, which gives us a map $H^*(BU(n)) \to H^*(X)$.
- Such a map is determined by $c_1(\eta), \ldots, c_n(\eta) \in H^*(X)$. These are the Chern classes of η.
- The $c_i(\eta)$ are obstructions to the trivializability of η.
- We want to generalize this framework to other cohomology theories.

Chern Classes

- Some calculations:
  - $H^*(BU(1)) \cong \mathbb{Z}[x]$.
  - $H^*(BU(n)) \cong H^*(BU(1)^n)^{\Sigma_n} \cong \mathbb{Z}[c_1, \ldots, c_n]$.
- Given a vector bundle η over X, we have a classifying map $X \to BU(n)$, which gives us a map $H^*(BU(n)) \to H^*(X)$.
- Such a map is determined by $c_1(\eta), \ldots, c_n(\eta) \in H^*(X)$. These are the Chern classes of η.
- The $c_i(\eta)$ are obstructions to the trivializability of η.
- We want to generalize this framework to other cohomology theories.

What is a Generalized Cohomology Theory?

- Take the axioms for ordinary cohomology and drop the dimension axiom.
- The coefficients no longer suffice to define the theory.
- We are interested in multiplicative theories (theories with cup products).
- $E^* : \mathrm{hTop}^{op} \to \mathrm{grRings}$. (Reverses arrows)

What is a Generalized Cohomology Theory?

- Take the axioms for ordinary cohomology and drop the dimension axiom.
- The coefficients no longer suffice to define the theory.
- We are interested in multiplicative theories (theories with cup products).
- $E^* : \mathrm{hTop}^{op} \to \mathrm{grRings}$. (Reverses arrows)

What is a Generalized Cohomology Theory?

- Take the axioms for ordinary cohomology and drop the dimension axiom.
- The coefficients no longer suffice to define the theory.
- We are interested in multiplicative theories (theories with cup products).
- $E^* : \mathrm{hTop}^{op} \to \mathrm{grRings}$. (Reverses arrows)

What is a Generalized Cohomology Theory?

- Take the axioms for ordinary cohomology and drop the dimension axiom.
- The coefficients no longer suffice to define the theory.
- We are interested in multiplicative theories (theories with cup products).
- $E^* : \mathrm{hTop}^{op} \to \mathrm{grRings}$. (Reverses arrows)

K-Theory

- $K^0(X) \cong \{[\eta] - [\zeta] \mid \eta, \zeta$ complex vector bundles over $X\}$.
- By extending $\oplus$ and $\otimes$ to formal differences of vector bundles, $K^0(X)$ becomes a ring.
- $K^{-1}(X) = K^0(\Sigma X)$.
- $K^n(X) \cong K^{n+2}(X)$ for all n. (Bott Periodicity)
- These groups can be pieced together to make a graded ring.

K-Theory

- $K^0(X) \cong \{[\eta] - [\zeta] \mid \eta, \zeta$ complex vector bundles over $X\}$.
- By extending $\oplus$ and $\otimes$ to formal differences of vector bundles, $K^0(X)$ becomes a ring.
- $K^{-1}(X) = K^0(\Sigma X)$.
- $K^n(X) \cong K^{n+2}(X)$ for all n. (Bott Periodicity)
- These groups can be pieced together to make a graded ring.

K-Theory

- $K^0(X) \cong \{[\eta] - [\zeta] \mid \eta, \zeta$ complex vector bundles over $X\}$.
- By extending $\oplus$ and $\otimes$ to formal differences of vector bundles, $K^0(X)$ becomes a ring.
- $K^{-1}(X) = K^0(\Sigma X)$.
- $K^n(X) \cong K^{n+2}(X)$ for all n. (Bott Periodicity)
- These groups can be pieced together to make a graded ring.

K-Theory

- $K^0(X) \cong \{[\eta] - [\zeta] \mid \eta, \zeta$ complex vector bundles over $X\}$.
- By extending $\oplus$ and $\otimes$ to formal differences of vector bundles, $K^0(X)$ becomes a ring.
- $K^{-1}(X) = K^0(\Sigma X)$.
- $K^n(X) \cong K^{n+2}(X)$ for all n. (Bott Periodicity)
- These groups can be pieced together to make a graded ring.

K-Theory

- $K^0(X) \cong \{[\eta] - [\zeta] \mid \eta, \zeta$ complex vector bundles over $X\}$.
- By extending $\oplus$ and $\otimes$ to formal differences of vector bundles, $K^0(X)$ becomes a ring.
- $K^{-1}(X) = K^0(\Sigma X)$.
- $K^n(X) \cong K^{n+2}(X)$ for all n. (Bott Periodicity)
- These groups can be pieced together to make a graded ring.

Generalized Chern Classes

- A complex-oriented cohomology theory E, is one in which we have a nice theory of Chern classes.
- In particular $E^*(BU(n)) \cong E_*[[c_1, \ldots, c_n]] \cong E^*(BU(1)^n)^{\Sigma_n}$. Where $E_* = E^*(S^0)$.
- The previous construction gives rise to "Chern classes" in $E^*(X)$.

Generalized Chern Classes

- A complex-oriented cohomology theory E, is one in which we have a nice theory of Chern classes.
- In particular $E^*(BU(n)) \cong E_*[[c_1, \ldots, c_n]] \cong E^*(BU(1)^n)^{\Sigma_n}$. Where $E_* = E^*(S^0)$.
- The previous construction gives rise to "Chern classes" in $E^*(X)$.

Generalized Chern Classes

- A complex-oriented cohomology theory E, is one in which we have a nice theory of Chern classes.
- In particular $E^*(BU(n)) \cong E_*[[c_1, \ldots, c_n]] \cong E^*(BU(1)^n)^{\Sigma_n}$. Where $E_* = E^*(S^0)$.
- The previous construction gives rise to "Chern classes" in $E^*(X)$.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

Formal Group Laws

- The $\otimes$ operation gives rise to a map $BU(1) \times BU(1) \to BU(1)$.
- Which gives rise to a comultiplication map $E^*(BU(1)) \cong E_*[[x]] \to E^*(BU(1) \times BU(1)) \cong E_*[[x, y]]$.
- Such a map defines a formal group law.
- Formal group laws appear in many branches of mathematics and are of great importance in homotopy theory.
- The associative, commutative and unital properties of $\otimes$ transform into coassociative, cocommutative and counital properties in cohomology.
- Since line bundles are invertible we obtain a cogroup structure in cohomology. This combined with the algebra structure make $E^*(BU(1))$ a Hopf algebra.

The Opposite Category

- Cooperations are unintuitive. Let's reverse our arrows again.
- Replace R by $\mathrm{Spec}(R) = \mathrm{Ring}(R, -) \in \mathrm{Set}^{\mathrm{Ring}}$. A functor isomorphic to $\mathrm{Spec}(R)$ for some R is called an affine scheme.
- The Yoneda Lemma tells us Spec gives an equivalence between Ring^{op} and the category of affine schemes.
- For finite dimensional spaces X we will study $X_E = \mathrm{Spec}(E^*(X))$, where we ignore the grading.
- Extend to infinite dimensional spaces X by setting $X_E = \mathrm{colim}\, Spec(E^*(X_i))$.

The Opposite Category

- Cooperations are unintuitive. Let's reverse our arrows again.
- Replace R by $\mathrm{Spec}(R) = \mathrm{Ring}(R, -) \in \mathrm{Set}^{\mathrm{Ring}}$. A functor isomorphic to $\mathrm{Spec}(R)$ for some R is called an affine scheme.
- The Yoneda Lemma tells us Spec gives an equivalence between Ring^{op} and the category of affine schemes.
- For finite dimensional spaces X we will study $X_E = \mathrm{Spec}(E^*(X))$, where we ignore the grading.
- Extend to infinite dimensional spaces X by setting $X_E = \mathrm{colim}\, Spec(E^*(X_i))$.

The Opposite Category

- Cooperations are unintuitive. Let's reverse our arrows again.
- Replace R by $\mathrm{Spec}(R) = \mathrm{Ring}(R, -) \in \mathrm{Set}^{\mathrm{Ring}}$. A functor isomorphic to $\mathrm{Spec}(R)$ for some R is called an affine scheme.
- The Yoneda Lemma tells us Spec gives an equivalence between Ring^{op} and the category of affine schemes.
- For finite dimensional spaces X we will study $X_E = \mathrm{Spec}(E^*(X))$, where we ignore the grading.
- Extend to infinite dimensional spaces X by setting $X_E = \mathrm{colim}\, Spec(E^*(X_i))$.

The Opposite Category

- Cooperations are unintuitive. Let's reverse our arrows again.
- Replace R by $\mathrm{Spec}(R) = \mathrm{Ring}(R, -) \in \mathrm{Set}^{\mathrm{Ring}}$. A functor isomorphic to $\mathrm{Spec}(R)$ for some R is called an affine scheme.
- The Yoneda Lemma tells us Spec gives an equivalence between Ring^{op} and the category of affine schemes.
- For finite dimensional spaces X we will study $X_E = \mathrm{Spec}(E^*(X))$, where we ignore the grading.
- Extend to infinite dimensional spaces X by setting $X_E = \mathrm{colim}\, Spec(E^*(X_i))$.

The Opposite Category

- Cooperations are unintuitive. Let's reverse our arrows again.
- Replace R by $\mathrm{Spec}(R) = \mathrm{Ring}(R, -) \in \mathrm{Set}^{\mathrm{Ring}}$. A functor isomorphic to $\mathrm{Spec}(R)$ for some R is called an affine scheme.
- The Yoneda Lemma tells us Spec gives an equivalence between Ring^{op} and the category of affine schemes.
- For finite dimensional spaces X we will study $X_E = \mathrm{Spec}(E^*(X))$, where we ignore the grading.
- Extend to infinite dimensional spaces X by setting $X_E = \mathrm{colim} Spec(E^*(X_i))$.

Identifying Representable Functors

- Example: $BU(1)_E(R) \cong \mathrm{Nil}(R)$.
- Now the $\otimes$ operation endows $\mathrm{Nil}(R)$ with an abelian group structure. This explains the term formal group.
- When E is integral cohomology we get the underlying additive group.
- When E is K-theory, the group operation takes (x, y) to $x + y + xy$.
- These are essentially the only formal group laws that can be expressed as polynomials as opposed to power series.

Identifying Representable Functors

- Example: $BU(1)_E(R) \cong \mathrm{Nil}(R)$.
- Now the $\otimes$ operation endows $\mathrm{Nil}(R)$ with an abelian group structure. This explains the term formal group.
- When E is integral cohomology we get the underlying additive group.
- When E is K-theory, the group operation takes (x, y) to $x + y + xy$.
- These are essentially the only formal group laws that can be expressed as polynomials as opposed to power series.

Identifying Representable Functors

- Example: $BU(1)_E(R) \cong \mathrm{Nil}(R)$.
- Now the $\otimes$ operation endows $\mathrm{Nil}(R)$ with an abelian group structure. This explains the term formal group.
- When E is integral cohomology we get the underlying additive group.
- When E is K-theory, the group operation takes (x, y) to $x + y + xy$.
- These are essentially the only formal group laws that can be expressed as polynomials as opposed to power series.

Identifying Representable Functors

- Example: $BU(1)_E(R) \cong \mathrm{Nil}(R)$.
- Now the $\otimes$ operation endows $\mathrm{Nil}(R)$ with an abelian group structure. This explains the term formal group.
- When E is integral cohomology we get the underlying additive group.
- When E is K-theory, the group operation takes (x, y) to $x + y + xy$.
- These are essentially the only formal group laws that can be expressed as polynomials as opposed to power series.

Identifying Representable Functors

- Example: $BU(1)_E(R) \cong \mathrm{Nil}(R)$.
- Now the $\otimes$ operation endows $\mathrm{Nil}(R)$ with an abelian group structure. This explains the term formal group.
- When E is integral cohomology we get the underlying additive group.
- When E is K-theory, the group operation takes (x, y) to $x + y + xy$.
- These are essentially the only formal group laws that can be expressed as polynomials as opposed to power series.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil} R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted “Burnside ring” for the trivial group.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil} R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted "Burnside ring" for the trivial group.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil} R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted "Burnside ring" for the trivial group.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil} R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted "Burnside ring" for the trivial group.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil} R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted “Burnside ring” for the trivial group.

Identifying Representable Functors II

- Example: $BU(n)_E(R) \cong (\mathrm{Nil}(R)^n)_{\Sigma_n}$.
- Think of this as specifying n elements, $\{x_i\}$, of $\mathrm{Nil}R$ (with repetition).
- The map $\oplus : BU(n)_E \times BU(m)_E \to BU(n+m)_E$ corresponds to the union of sets with multiplicity.
- In particular, the map $\oplus : BU(n)_E \times BU(1)_E \to BU(n+1)_E$ corresponds to adjoining an extra 0.
- The map $\otimes : BU(n)_E \times BU(m)_E \to BU(nm)_E$ corresponds to the map $\{x_i\} \times \{y_j\} \to \{x_i +_F y_j\}$.
- This makes $(BU \times \mathbb{Z})_E$ into a twisted “Burnside ring” for the trivial group.

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a “shift”.

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a "shift".

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a "shift".

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a "shift".

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a “shift”.

The polynomial perspective

- Can also think of the elements of $BU(n)_E(R)$ as specifying monic degree n polynomials, i.e. $f(z) = z^n - c_1 z^{n-1} + \cdots \pm c_n = \prod(z - x_i)$.
- Direct sum becomes multiplication of polynomials and tensor product performs the above operation on roots.
- We can use this to determine $c_i(\eta \otimes \zeta)$, where η and ζ are arbitrary vector bundles. (Implemented in Mathematica)
- Adjoining a trivial bundle corresponds to multiplying by z.
- Taking colimits, we invert z and see that $BU_E(R)$ corresponds to polnomials of the form $f(t) = 1 + c_1 t + \cdots + c_n t^n$ where n is nonnegative ($t = 1/z$).
- Direct sum still corresponds to multiplication of polynomials but the tensor product involves a “shift”.

The polynomial perspective II

- This is almost the lambda scheme, $\Lambda(-)$. Which assigns to a ring R the set $1 + tR[[t]]$ with an unusual ring structure. Abelian group structure is defined by multiplication of power series, multiplicative structure is the same as the above if we replace $x +_F y$ with xy.
- We actually have the Cartier dual of the lambda scheme, which no longer takes values in rings, but rngs. The multiplicative structure is twisted by the formal group law.
- There is an isomorphism $\mathbb{W} \cong \Lambda$ which induces isomorphisms on the duals, $\hat{\mathbb{W}} \cong \hat{\Lambda}$.
- It turns out that $\hat{\mathbb{W}}$ represents the curves functor for formal groups.

The polynomial perspective II

- This is almost the lambda scheme, $\Lambda(-)$. Which assigns to a ring R the set $1 + tR[[t]]$ with an unusual ring structure. Abelian group structure is defined by multiplication of power series, multiplicative structure is the same as the above if we replace $x +_F y$ with xy.
- We actually have the Cartier dual of the lambda scheme, which no longer takes values in rings, but rngs. The multiplicative structure is twisted by the formal group law.
- There is an isomorphism $\mathbb{W} \cong \Lambda$ which induces isomorphisms on the duals, $\hat{\mathbb{W}} \cong \hat{\Lambda}$.
- It turns out that $\hat{\mathbb{W}}$ represents the curves functor for formal groups.

The polynomial perspective II

- This is almost the lambda scheme, $\Lambda(-)$. Which assigns to a ring R the set $1 + tR[[t]]$ with an unusual ring structure. Abelian group structure is defined by multiplication of power series, multiplicative structure is the same as the above if we replace $x +_F y$ with xy.
- We actually have the Cartier dual of the lambda scheme, which no longer takes values in rings, but rngs. The multiplicative structure is twisted by the formal group law.
- There is an isomorphism $\mathbb{W} \cong \Lambda$ which induces isomorphisms on the duals, $\hat{\mathbb{W}} \cong \hat{\Lambda}$.
- It turns out that $\hat{\mathbb{W}}$ represents the curves functor for formal groups.

The polynomial perspective II

- This is almost the lambda scheme, $\Lambda(-)$. Which assigns to a ring R the set $1 + tR[[t]]$ with an unusual ring structure. Abelian group structure is defined by multiplication of power series, multiplicative structure is the same as the above if we replace $x +_F y$ with xy.
- We actually have the Cartier dual of the lambda scheme, which no longer takes values in rings, but rngs. The multiplicative structure is twisted by the formal group law.
- There is an isomorphism $\mathbb{W} \cong \Lambda$ which induces isomorphisms on the duals, $\hat{\mathbb{W}} \cong \hat{\Lambda}$.
- It turns out that $\hat{\mathbb{W}}$ represents the curves functor for formal groups.

Summary

- The schemes X_E, for the above X represent interesting functors.
- The representing functors are a convenient way of packaging all of the structure in $E^*(X)$.
- Using these constructions we have discovered families of generalized Lambda schemes and Witt schemes parametrized by formal group laws.

Summary

- The schemes X_E, for the above X represent interesting functors.
- The representing functors are a convenient way of packaging all of the structure in $E^*(X)$.
- Using these constructions we have discovered families of generalized Lambda schemes and Witt schemes parametrized by formal group laws.

Summary

- The schemes X_E, for the above X represent interesting functors.
- The representing functors are a convenient way of packaging all of the structure in $E^*(X)$.
- Using these constructions we have discovered families of generalized Lambda schemes and Witt schemes parametrized by formal group laws.

Future Directions

- Replace $BU \times \mathbb{Z}$ with $\Omega^\infty F$, where F is the ring spectrum associated to a complex oriented theory.
- Apply to the study of $Im(J)$ and $Coker(J)$.
- Keep better track of how these cohomology rings parametrize unstable operations.

Future Directions

- Replace $BU \times \mathbb{Z}$ with $\Omega^{\infty} F$, where F is the ring spectrum associated to a complex oriented theory.
- Apply to the study of $Im(J)$ and $Coker(J)$.
- Keep better track of how these cohomology rings parametrize unstable operations.

Future Directions

- Replace $BU \times \mathbb{Z}$ with $\Omega^\infty F$, where F is the ring spectrum associated to a complex oriented theory.
- Apply to the study of $Im(J)$ and $Coker(J)$.
- Keep better track of how these cohomology rings parametrize unstable operations.