# For Complex Orientations Preserving Power Operations, $p$-Typicality is Atypical

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# *MU*

*MU* can be taken as a nexus in stable homotopy theory.

- There is a spectrum *MU* satisfying:

$$\pi_n MU \cong \{\text{Complex cobordism classes of n-manifolds}\}.$$

- There is a spectral sequence (ANSS)

$$Ext^{*,*}_{MU_* MU}(MU_*, MU_*) \implies \pi_* S.$$

- *MU* serves as a conduit between the theory of formal group laws and stable homotopy theory.

# The Adams Novikov Spectral Sequence

- The determination of

$$E_2^{*,*} = Ext^{*,*}_{MU_* MU}(MU_*, MU_*)$$

  is an algebraic problem which only requires data from the theory of formal group laws.
- Philosophy: The differentials and extensions in this spectral sequence yielding $\pi_* S$ can always be determined by using $E_\infty$ structures.
- One can try to use constructions in the theory of formal group laws, to organize the above calculation into digestible pieces.
- Before saying more about this, let me review some basics of homotopy theory.

# Cohomology theories and spectra

So what is a spectrum?

- Generalized cohomology theories are realized by spectra.
- This permits manipulating cohomology theories like algebraic objects (modules, rings, etc.).
- We have a functor

$$\Sigma^{\infty}_{+} : hTop \to hSpectra.$$

- We also have constructions in spectra that mimic those of topological spaces (Cartesian products, smash products, fibrations).
- Additive approximation to the homotopy category of spaces.

# Cohomology theories and spectra II

- Maps

$$\Sigma^i E \to \Sigma^j F$$

correspond to natural transformations

$$E^*(-) \to F^{*+j-i}(-)$$

- If $E$ is a ring spectrum, then $E^*(-)$ takes spaces to graded rings, the multiplicative structure comes from maps

$$E \wedge E \to E.$$

# Formal group laws

- The tensor product of complex line bundles induces a map on classifying spaces:

$$BU(1) \times BU(1) \to BU(1).$$

- This induces maps the other way in cohomology:

$$\pi_* MU[\![z]\!] \cong MU^*(BU(1)) \xrightarrow{\Delta} MU^*(BU(1) \times BU(1)) \cong \pi_* MU[\![x, y]\!].$$

- The associative, unital, and symmetric properties of the tensor product make

$$\Delta(z) = F(x, y) = x +_F y \in \pi_* MU[\![x, y]\!]$$

into a formal group law over $\pi_* MU$.

# Formal group laws II

- $F(x, 0) = 0 = F(0, y)$.
- $F(x, y) = F(y, x)$.
- $F(F(x, y), z) = F(x, F(y, z))$.
- Example ($\mathbb{G}_a$): $F(x, y) = x + y$.
- Example ($\mathbb{G}_m$): $F(x, y) = x + y + xy$.

# Lazard's Theorem

## Theorem (Lazard)

*The ring* $L \cong \mathbb{Z}[u_1, u_2, \dots]$ *can be equipped with a universal formal group law* $U(x, y) \in L[\![x, y]\!]$. *In other words, if* $G(x, y)$ *is a formal group law* $R$, *then there is a unique map*

$$f : L \to R$$

*such that*

$$f_* U(x, y) = G(x, y).$$

# Quillen's Theorem

## Theorem (Quillen)

*The universal map from L to $\pi_* MU$ is an isomorphism.*

## Corollary

$$\pi_* MU \cong \mathbb{Z}[u_1, u_2, \dots], \ |u_i| = 2i.$$

## Theorem (Adams)

*A complex orientation $MU \to E$ is a map of ring spectra that determines a formal group law in $E^*(BU(1) \times BU(1))$ under the above isomorphism.*

# $p$-typical formal group laws

## Definition

A formal group law $F$ is $p$-typical if the $p$-fold formal sum $[p]_F(z) = z +_F z +_F \cdots +_F z$ can be expressed as

$$[p]_F(z) = \sum_{n \geq 0}^{F} a_n z^{p^n}.$$

# $p$-typical formal group laws

- Alternatively, if $F$ is defined over a torsion free ring this is equivalent to requiring

$$\log_F(x) = \sum_{n \geq 0} b_n x^{p^n}$$

where

$$x +_F y = \log_F^{-1}(\log_F(x) + \log_F(y)).$$

# The Brown-Peterson spectrum

- Applying a standard construction in the theory of formal group laws, Quillen was able to construct an idempotent map:

$$MU_{(p)} \xrightarrow{\varepsilon} MU_{(p)}$$

  that classified a universal $p$-typical formal group law.
- The "image" of this map $MU_{(p)}$ is the Brown-Peterson ring spectrum $BP$. It satisfies a $p$-typical analogue of the universal property for $MU$.

# $E_\infty$ ring spectra

- So far, everything has been up to homotopy.
- Would like our homotopy commutative and associative multiplication to satisfy some additional coherence data.
- For example, given a cohomology class

$$f : X \to F \in F^0(X),$$

take its $p$th power and then multiply:

$$\mu^{p-1} \circ f^{\wedge p} : X^{\wedge p} \to F^{\wedge p} \to F.$$

We would like the composite to be $\Sigma_p$ equivariant.

# $E_\infty$ ring spectra II

- If $F$ is suitably commutative then, we can extend our multiplication over a Borel construction before passing to homotopy:

$$D_p\mu \circ D_p f : E\Sigma_{p,+} \wedge_{\Sigma_p} X^{\wedge p} \to E\Sigma_{p,+} \wedge_{\Sigma_p} F^{\wedge p} \to F.$$

- $E\Sigma_{p,+} \wedge_{\Sigma_p} X^{\wedge p}$ is the homotopical analogue of $X^{\wedge p}/\Sigma_p$.
- If $F$ admits such structure maps in a suitably compatible way then $F$ is an $E_\infty$ ring spectrum.
- Such spectra have a product that can be made strictly unital, associative, and commutative.
- This structure is enough to perform many algebraic constructions in spectra (EKMM).

# $H_\infty$ ring spectra

- The definition of $E_\infty$ ring spectra predated most of the applications by about 20 years.
- For most applications it sufficed to have the above coherence data hold in the homotopy category.
- Such a spectrum is an $H_\infty$ ring spectrum.
- This data is exactly enough to define a nice theory of power operations satisfying analogues of the Adem relations.
- Examples: $H\mathbb{Z}/p, K, MU$.

# An example due to Kraines-Lada

- After passing to the homotopy category an $E_\infty$ ring spectrum becomes an $H_\infty$ ring spectrum.
- Does the converse hold?
- No. Applying $\Sigma^\infty_+$ to the counterexample to the transfer conjecture constructed by Kraines and Lada gives an example of a ring spectrum which is not $E_3$, yet this multiplicative structure defines an $H_\infty$ structure in the homotopy category.

# A well known conjecture

- *MU* is a very beautiful spectrum with a canonical $E_\infty$ ring structure. The Brown-Peterson spectrum is also quite beautiful and closely related *MU*.

## Conjecture

*BP is* $E_\infty$.

- Some partial results:
  - (Basterra-Mandell) *BP* is $E_4$.
  - (Richter) *BP* is $2(p^2 + p + 1)$ homotopy-commutative.
  - (Goerss/Lazarev) *BP* and *many* of its derivatives are $E_1 = A_\infty$-spectra under *MU* (in many ways).

# Our result

**Theorem (Johnson-Noel)**

*If $p$ is a prime smaller than 17, then Quillen's map $r : MU \to BP$ is not a map of $H_\infty$ ring spectra.*

# The more general result

## Theorem (Johnson-Noel)

*Suppose $f : MU \to E$ is a map of $H_\infty$ ring spectra satisfying:*

- *The map $f$ factors through $BP$*

  $$MU \to MU_{(p)} \to BP \to E,$$

  *(in other words, the formal group law associated to $E$ is $p$-typical)*
- *the map $f$ makes $E_*$ into a Landweber exact theory,*
- *the above prime $p$ is smaller than 17,*
- *then $\pi_* E$ is a $\mathbb{Q}$-algebra $\implies$ $E$ is a generalized rational Eilenberg-Maclane spectra ($E$ is a retract of $H\mathbb{Q} \wedge E$).*

# Some notation

- Fix an isomorphism

$$MU^*(BU(1) \times BC_p) \cong \pi_* MU[\![x, \xi]\!]/([p]_{MU}\xi)$$

- Let $\chi = \prod_{i=1}^{p-1} [i]_{MU}\xi$ denote the *MU* Euler class of the reduced regular representation of $C_p$.
- Define $a_i(\xi)$ by

$$P_{C_p}(x) = \prod_{i=0}^{p-1} ([i]_{MU}\xi +_{MU} x) = a_0(\xi)x + a_1(\xi)x^2 + a_2(\xi)x^3 + \cdots$$

# Algebraic interpretation

**Theorem**

*The map $r : MU \to BP$ is an $H_\infty$ map if and only if $r \circ \overline{P_{C_p,MU}}$ defines $p$-typical formal group law over $BP^{BC_p}[\chi^{-1}]$.*

$$\begin{array}{ccc}
MU^{2*}(BU(1)) & \xrightarrow{\overline{P_{C_p,MU}}} & MU^{BC_p,2*}[\chi^{-1}](BU(1)) \\
\Big\downarrow r & & \Big\downarrow r \\
BP^{*}(BU(1)) & \xrightarrow{\overline{P_{C_p,BP}}} & BP^{BC_p,2*}[\chi^{-1}](BU(1))
\end{array}$$

# Critical Lemmas

## Theorem (Quillen)

*If $x \in MU^{-2q}(X)$ and $m \gg 0$ then*

$$\chi^{m+q} P_{C_p} x = \chi^q \sum_{|\alpha|=m} a_\alpha s_\alpha(x).$$

## Theorem (Adams-Novikov)

$$s_\alpha[\mathbb{C}P^n] = (b_0 + b_1 + b_2 + \cdots)_\alpha^{-(n+1)}[\mathbb{C}P^{n-|\alpha|'}]$$

# Formula

$$
\begin{aligned}
MC_n(\xi) &= \chi^{2n} P_{C_p}[\mathbb{C}P^n] \\
&= \chi^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left( \left( \textstyle\sum_{i\geq 0} a_i z^i \right)^{-(n+1)} \right) [z^k].
\end{aligned}
$$

# Computational Results

- When $p = 2$, $MC_2(\xi) = \left(v_1^6 + v_2^2\right)\xi^6 + O(\xi^7)$.
- When $p = 3$, $MC_4(\xi) = 2v_1^9\xi^{22} + O(\xi^{23})$.
- When $p = 5$, $MC_8(\xi) = 3v_1^{16}\xi^{88} + O(\xi^{89})$.
- When $p = 7$, $MC_{12}(\xi) = 4v_1^{22}\xi^{192} + O(\xi^{193})$.
- When $p = 11$, $MC_{20}(\xi) = 9v_1^{34}\xi^{520} + O(\xi^{521})$.
- When $p = 13$, $MC_{24}(\xi) = 11v_1^{40}\xi^{744} + O(\xi^{745})$.

# Pattern

We have the following pattern: In the computed range, for $p \geq 5$ up to multiplication by units:

$$MC_{2(p-1)}(\xi) = v_1^{3p+1}\xi^{5p^2-8p+3} + O(\xi^{5p^2-8p+4}).$$

# Summary

- Each of the previous calculations needs to be 0 mod $\langle p \rangle \, \xi$.
- In *BP* they are not zero because the leading term is not divisible by $p$.
- These are the first of many obstructions to an $E_\infty$ map.
- In our additional examples, these obstructions will not vanish unless the target ring is the 0-ring.

# Possibilities

- Could there be an $E_\infty$ orientation for higher primes?
  - That would be confusing.
  - $p$-typicality is not a natural notion
- Geometry vs. Algebra
- Perhaps better to build $E_\infty$ spectrum models for the algebra independently.

# Some notation

- Let $\alpha = (\alpha_0, \alpha_1, \dots)$ be a multi-index
- Given an infinite list of variable $a_0, a_1, a_2, \dots,$ we set

$$a^\alpha = a_0^{\alpha_0} a_1^{\alpha_1} \dots .$$

- Given a formal power series $b$ in $b_0, b_1, b_2, \dots$ we set:

$$b_\alpha = \text{Coefficient of } b_0^{\alpha_0} b_1^{\alpha_1} \dots .$$

- $|\alpha| = \sum_{i \geq 1} \alpha_i$.
- $|\alpha|' = \sum_{i \geq 1} i\alpha_i$.