For Complex Orientations Preserving Power Operations, *p*-Typicality is Atypical

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- *MU* serves as a conduit between the theory of formal group laws and stable homotopy theory.

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- One can try to use constructions in the theory of formal group laws, to organize the above calculation into digestible pieces.
- Before saying more about this, let me review some basics of homotopy theory.

Cohomology theories and spectra

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- Additive approximation to the homotopy category of spaces.

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- If E is a ring spectrum, then $E^*(-)$ takes spaces to graded rings, the multiplicative structure comes from maps
$$E \wedge E \to E.$$

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- The associative, unital, and symmetric properties of the tensor product make

$$\Delta(z) = F(x, y) = x +_F y \in \pi_* MU[\![x, y]\!]$$

into a formal group law over $\pi_* MU$.

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- Example ($\mathbb{G}_m$): $F(x, y) = x + y + xy$.

Lazard's Theorem

Theorem (Lazard)

The ring $L \cong \mathbb{Z}[u_1, u_2, \ldots]$ can be equipped with a universal formal group law $U(x, y) \in L[\![x, y]\!]$. In other words, if $G(x, y)$ is a formal group law R, then there is a unique map

$$f : L \to R$$

such that

$$f_* U(x, y) = G(x, y).$$

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Theorem (Adams)

A complex orientation $MU \to E$ is a map of ring spectra that determines a formal group law in $E^*(BU(1) \times BU(1))$ under the above isomorphism.

p-typical formal group laws

Definition

A formal group law F is p-typical if the p-fold formal sum $[p]_F(z) = z +_F z +_F \cdots +_F z$ can be expressed as

$$[p]_F(z) = \sum_{n \geq 0}^{F} a_n z^{p^n}.$$

p-typical formal group laws

- Alternatively, if F is defined over a torsion free ring this is equivalent to requiring

$$\log_F(x) = \sum_{n \geq 0} b_n x^{p^n}$$

where

$$x +_F y = \log_F^{-1}(\log_F(x) + \log_F(y)).$$

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that classified a universal p-typical formal group law.
- The "image" of this map $MU_{(p)}$ is the Brown-Peterson ring spectrum BP. It satisfies a p-typical analogue of the universal property for MU.

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- For example, given a cohomology class

$$f : X \to F \in F^0(X),$$

take its pth power and then multiply:

$$\mu^{p-1} \circ f^{\wedge p} : X^{\wedge p} \to F^{\wedge p} \to F.$$

We would like the composite to be Σ_p equivariant.

E_∞ ring spectra II

- If F is suitably commutative then, we can extend our multiplication over a Borel construction before passing to homotopy:

$$D_p\mu \circ D_p f : E\Sigma_{p,+} \wedge_{\Sigma_p} X^{\wedge p} \to E\Sigma_{p,+} \wedge_{\Sigma_p} F^{\wedge p} \to F.$$

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- Such spectra have a product that can be made strictly unital, associative, and commutative.
- This structure is enough to perform many algebraic constructions in spectra (EKMM).

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- Examples: $H\mathbb{Z}/p, K, MU$.

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- After passing to the homotopy category an E_∞ ring spectrum becomes an H_∞ ring spectrum.
- Does the converse hold?
- No. Applying Σ^∞_+ to the counterexample to the transfer conjecture constructed by Kraines and Lada gives an example of a ring spectrum which is not E_3, yet this multiplicative structure defines an H_∞ structure in the homotopy category.

A well known conjecture

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- Some partial results:
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  - (Richter) *BP* is $2(p^2 + p + 1)$ homotopy-commutative.
  - (Goerss/Lazarev) *BP* and *many* of its derivatives are $E_1 = A_\infty$-spectra under *MU* (in many ways).

Our result

Theorem (Johnson-Noel)

If p is a prime smaller than 17, then Quillen's map $r : MU \to BP$ is not a map of H_∞ ring spectra.

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- *the map f makes E_* into a Landweber exact theory,*
- *the above prime p is smaller than 17,*
- *then $\pi_* E$ is a $\mathbb{Q}$-algebra $\implies$ E is a generalized rational Eilenberg-Maclane spectra (E is a retract of $H\mathbb{Q} \wedge E$).*

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- Define $a_i(\xi)$ by

$$P_{C_p}(x) = \prod_{i=0}^{p-1} ([i]_{MU}\xi +_{MU} x) = a_0(\xi)x + a_1(\xi)x^2 + a_2(\xi)x^3 + \cdots$$

Algebraic interpretation

Theorem

The map $r : MU \to BP$ is an H_∞ map if and only if $r \circ \overline{P_{C_p,MU}}$ defines p-typical formal group law over $BP^{BC_p}[\chi^{-1}]$.

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$$\begin{array}{ccc} MU^{2*}(BU(1)) & \xrightarrow{\overline{P_{C_p,MU}}} & MU^{BC_p,2*}[\chi^{-1}](BU(1)) \\ \downarrow r & & \downarrow r \\ BP^{*}(BU(1)) & \xrightarrow{\overline{P_{C_p,BP}}} & BP^{BC_p,2*}[\chi^{-1}](BU(1)) \end{array}$$

Critical Lemmas

Theorem (Quillen)

If $x \in MU^{-2q}(X)$ *and* $m \gg 0$ *then*

$$\chi^{m+q} P_{C_p} x = \chi^q \sum_{|\alpha|=m} a_\alpha s_\alpha(x).$$

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Theorem (Adams-Novikov)

$$s_\alpha[\mathbb{C}P^n] = (b_0 + b_1 + b_2 + \cdots)_\alpha^{-(n+1)}[\mathbb{C}P^{n-|\alpha|'}]$$

Formula

$$
\begin{aligned}
MC_n(\xi) &= \chi^{2n} P_{C_p}[\mathbb{C}P^n] \\
&= \chi^{2n+1} \sum_{k=0}^{n} r_*[\mathbb{C}P^{n-k}] \cdot \left(\left(\sum_{i \geq 0} a_i z^i \right)^{-(n+1)} \right) [z^k].
\end{aligned}
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- When $p = 13$, $MC_{24}(\xi) = 11v_1^{40}\xi^{744} + O(\xi^{745})$.

Pattern

We have the following pattern: In the computed range, for $p \geq 5$ up to multiplication by units:

$$MC_{2(p-1)}(\xi) = v_1^{3p+1}\xi^{5p^2-8p+3} + O(\xi^{5p^2-8p+4}).$$

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- In BP they are not zero because the leading term is not divisible by p.
- These are the first of many obstructions to an E_∞ map.
- In our additional examples, these obstructions will not vanish unless the target ring is the 0-ring.

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- Perhaps better to build E_∞ spectrum models for the algebra independently.

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- $|\alpha|' = \sum_{i \geq 1} i\alpha_i$.