On the Morava *K*-Theory of Some Infinite Loop Spaces

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Introduction

- Revisiting the theory of infinite loop spaces.

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- Recall some classic results and place them inside of a modern framework.
- Identify the edge our understanding and some questions that lie beyond that edge.

Infinite Loop Spaces

Definition

An infinite loop space $X = X_0$, is a sequence $\{X_i\}_{i \geq 0}$ based spaces with homeomorphisms

$$f_{i+1} : \Omega X_{i+1} := Top_*(S^1, X_{i+1}) \to X_i.$$

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- Implicit abuse of notation.

Example

BU admits two different infinite loop space structures.

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$$\begin{aligned}\pi_0 Top(X, BU^{\oplus}) &\cong \widetilde{K}^0(X) \\ &\cong \{[V] - [W] \mid V, W \text{ vector bundles over } X \\ &\qquad \text{w/ } dim(V) = dim(W)\}/\sim .\end{aligned}$$

Group under direct sum.

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Group under tensor product using distributivity:

$$([V_1]-[W_1])\otimes([V_2]-[W_2]) = [(V_1\otimes V_2)\oplus(W_1\otimes W_2)] \\ -[(V_1\otimes W_2)\oplus(V_2\otimes W_1)].$$

Equivalence of Categories

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- This functor restricts to a suitable equivalence of categories
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- So, if E and F are connective spectra, E^0F can be identified with homotopy classes of infinite loop maps $[\Omega^\infty E, \Omega^\infty F]$.
- The two-sided bar construction provides an inverse functor for this equivalence:
$$B(\Sigma^\infty, \Omega^\infty\Sigma^\infty, -) : \mathit{InfLoop} \to \mathit{Spectra}.$$

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- This adjunction defines a monad $Q = \Omega^\infty\Sigma^\infty$ whose algebras are infinite loop spaces.
- The evaluation natural transformation

$$\varepsilon : \Sigma^\infty\Omega^\infty \to Id$$

and the unit map

$$e : Id \to \Omega^\infty\Sigma^\infty$$

define the Q-algebra structure on $\Omega^\infty E$.

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- Here $B_n(\Sigma^\infty, Q, \Omega^\infty E)$ is the suspension spectrum

$$\Sigma^\infty Q^n \Omega^\infty E.$$

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- Alternatively, we can consider E_∞ spaces.

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- $\bar{\mathbb{P}}$ is a monad whose algebras are E_∞ spaces.

Questions

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3. When can we identify the image of the above map with those maps that commute with transfer homomorphisms?

Answer 1: Then

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- BSU_p.
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- For future reference note that $bso = P_{\geq 2}KO$ and $bsu = P_{\geq 4}KU$.
- Note $L_{K(1)}KO = KO_p$ and $L_{K(1)}KU = KU_p$. So these spaces are connected covers of $K(n)$-local spectra.

Answer 2: Then

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is an injection when:
$E = F$ is an i-connected cover of p-complete real or complex K-theory.

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We can identify the image of

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- One might conjecture that similar results hold for $K(n)$-local spectra.

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- There are functors $\phi_n : \mathit{Top}_* \to L_{T(n)}\mathit{Spectra}$.
- $T(n) = F[v_n^{-i}]$, where F is some type n finite spectrum.
- Satisfying $L_{K(n)} = L_{K(n)}\phi_n\Omega^\infty$.

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A space $\Omega^\infty E$ admits a unique infinite loop space structure if:
$\Omega^\infty E$ is $T(n-1)$-acyclic and suitably connected.
$\Omega^\infty E$ can be constructed from finite type *n spaces*.

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is an injection when:
F is $K(n)$-local and some spectral sequences converge.

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$K(n)$-homology of Eilenberg-MacLane spaces

Theorem (Ravenel-Wilson)

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Corollary

Eilenberg-Maclane spectra are (strongly) $K(n)$-acyclic.

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- We have $E(n-1)_*C_nF \cong 0 \cong K(n)_*L_{n-1}F$.

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- In general, these maps do not induce $K(n)$-equivalences after applying Ω^∞.

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Theorem (Ravenel-Wilson)

There is an elegant, although non-trivial, description of $K(n)_*K(\mathbb{Z}, j)$ *for* $0 \leq j \leq n+1$.

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Theorem (Hopkins-Ravenel-Wilson)

Suppose E is a spectrum with finite homotopy, then

$$K(n)_*(\Omega^\infty E) \cong \bigotimes_{i=0}^{n} K(n)_*(K(\pi_i E, i).$$