On a nilpotence conjecture of J.P. May

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May's conjecture

Conjecture (May - 1986)

Suppose that R is a ring spectrum with power operations, i.e., an H_∞-ring spectrum, and

$$x \in \ker(\pi_* R \to H_*(R;\mathbb{Z})).$$

Then x is nilpotent., i.e., $x^n = 0$ for $n \gg 0$.

- When $R = S \implies$ Nishida's nilpotence theorem.
- In contrast to the nilpotence theorem (Devinatz-Hopkins-Smith):
  We do not need to know about $MU_* R$, but now require R to be H_∞.

E_∞/H_∞-rings

- An E_∞-ring spectrum R has extended power maps:

$$\mu_n : E\Sigma_{n+} \wedge_{\Sigma_n} R^n \to R$$

- The $\{\mu_n\}_{n\geq 0}$ are required to fit into some commutating diagrams.
- All of these diagrams are in Spectra.
- Analogous definition in *ho*(Spectra) yields H_∞-ring spectra.
- Obtain forgetful functor:

$$U : ho(E_\infty\text{–Spectra}) \to H_\infty\text{–Spectra}.$$

- R is $E_\infty \simeq$ comm. S-algebra $\Longrightarrow R$ is H_∞.

Examples of E_∞/H_∞-rings

Let X be a simply connected space of finite type.

k	Spaces	$E_\infty\mathrm{Mod}_{Hk}$	$H_\infty\mathrm{Mod}_{Hk} \subset k_*\mathrm{v.s.}$
$\mathbb{Q}$	$X_{\mathbb{Q}}$	$H\mathbb{Q}^X \simeq C^*(X;\mathbb{Q})$	$H^*(X;\mathbb{Q})$
$\overline{\mathbb{F}}_p$	X_p	$H\overline{\mathbb{F}}_p^X \simeq C^*(X;\overline{\mathbb{F}}_p)$	$H^*(X;\overline{\mathbb{F}}_p)$

- Sullivan theory says we can recover $X_{\mathbb{Q}}$ from the E_∞ algebra $H\mathbb{Q}^X$.
- Only get $H^*(X;\mathbb{Q}) \in \mathbb{Q}\text{-}\mathcal{CAlg}$.
- Mandell's theory says we can recover X_p from the E_∞-algebra $H\overline{\mathbb{F}}_p^X$.
- Only get $H^*(X;\overline{\mathbb{F}}_p)$ as an unstable algebra over Steenrod algebra.

Main result

Theorem (Mathew-Naumann-N.)

Suppose that R is an H_∞-ring spectrum. Suppose that $x \in \pi_* R$ has nilpotent image in $H_*(R;\mathbb{Q})$ and $H_*(R;\mathbb{F}_p)$ for $\forall$ primes p. Then x is nilpotent.

Corollary

May's conjecture: Suppose that R is an H_∞-ring spectrum and

$$x \in \ker(\pi_* R \to H_*(R;\mathbb{Z})).$$

Then x is nilpotent.

Proof reduction

Theorem (Hopkins-Smith)

Suppose that R is a ring spectrum and that $x \in \pi_* R$ has nilpotent image in

1. $H_*(R;\mathbb{Q})$
2. $H_*(R;\mathbb{F}_p)$ *for* $\forall$ *primes* p
3. *and* $K(n)_*(R)$ *for* $\forall$ *primes* p *and* $n > 0$.

Then x is nilpotent.

Theorem (Mathew-Naumann-N.)

If R is H_∞ then condition (1) *imply condition* (3) *and therefore* (1) *and* (2) *imply the main theorem.*

Proof of thm

- Since x has nilpotent image $H_*(R;\mathbb{Q}) \cong \pi_* R \otimes \mathbb{Q}$, $\exists m$ s.t. x^m is torsion.
- $y := x^m$ is nilpotent $\iff$ x is nilpotent.
- WTS that our torsion y has nilpotent image in $K(n)_* R$.
- Hurewicz map factors

$$\pi_* R \to \pi_* L_{K(n)}(E_n \wedge R) \to K(n)_*(R).$$

- $\mathrm{Im}(y) \in \pi_* L_{K(n)}(E_n \wedge R)$ is nilpotent $\implies$ $\mathrm{Im}(y) \in K(n)_*(R)$ is nilpotent $\implies$ the main theorem.

Proof of thm II

- $\pi_* L_{K(n)}(E_n \wedge R)$ is p-local.
- Assume $y \in \pi_* L_{K(n)}(E_n \wedge R)$ is p^m-torsion.
- We then show that $y^{p+1} \in \pi_* L_{K(n)}(E_n \wedge R)$ is p^{m-1}-torsion.
- So $y^{(p+1)^m} = 0 \in \pi_* L_{K(n)}(E_n \wedge R)$ and the theorem follows.
- Argument uses analogues of ψ/θ-operations and the formula:

$$y^p = \psi(y) + p\theta(y).$$

- This depends on work of Rezk and Strickland.

String manifolds

Recall that a *String* structure on a smooth manifold M is

$$\begin{array}{ccccc} & & & & BString = BO\langle 8\rangle \\ & & & \nearrow & \downarrow \\ M & \xrightarrow[\tau]{} & BO(n) & \longrightarrow & BO \end{array}$$

- Similar definitions for *Spin*-manifolds, *SO*-manifolds, ...
- $\rightsquigarrow E_\infty$ Thom spectra $MString, MSO, \ldots$
- $\pi_* MString \cong \Omega^*_{String}, \pi_* MSpin \cong \Omega^*_{Spin} \ldots$
- Forgetful maps:

$$MString \to MSpin \to MSO \to MO$$

String bordism

Theorem (Mathew-Naumann-N.)

For a String manifold M. The following are equivalent:

1. *For $n \gg 0$, $M^{\times n}$ bounds a String manifold.*
2. *M bounds an oriented manifold.*
3. *The Stiefel-Whitney and Pontryagin numbers of M vanish.*

- (2) $\iff$ (3) is classical.
- Claim is equivalent to

$$\mathrm{Nil}(\pi_* MString) = \ker(\pi_* MString \to \pi_* MSO).$$

- We will show these are both are equal to

$$\ker(\pi_* MString \to H_*(MString; \mathbb{Z})).$$

$\mathrm{Nil}(\pi_* MString)$

- May's conjecture $\implies$

$$\ker(\pi_* MString \to H_*(MString;\mathbb{Z})) \subseteq \mathrm{Nil}(\pi_* MString).$$

- Always have

$$\mathrm{Im}(\mathrm{Nil}(\pi_* MString)) \subseteq \mathrm{Nil}(H_*(MString;\mathbb{Z})).$$

- We will show

$$\mathrm{Nil}(H_*(MString;\mathbb{Z})) \subseteq \mathrm{Nil}(H_*(MSO;\mathbb{Z})) = 0.$$

$$\implies \ker(\pi_* MString \to H_*(MString;\mathbb{Z})) = \mathrm{Nil}(\pi_* MString).$$

$H_*(MString;\mathbb{Z})$

We will show:

$$\begin{array}{c} H_*(MString;\mathbb{Z}) \\ \big\downarrow \\ H_*(MSO;\mathbb{Z}) \end{array}$$

$$\begin{array}{ccc} \pi_* MString & \longrightarrow & H_*(MString;\mathbb{Z}) \\ \big\downarrow & & \big\downarrow \\ \pi_* MSO & \hookrightarrow & H_*(MSO;\mathbb{Z}) \end{array}$$

$$\implies \ker(\pi_* MString \to \pi_* MSO) = \ker(\pi_* MString \to H_*(MString;\mathbb{Z})).$$

So want to analyze $H_*(MString;\mathbb{Z})$.

Thom isomorphisms

- Have multiplicative isos $H_*(MG;k) \cong H_*(BG;k)$ for $G = SO, Spin, String$.
- So we need to analyze $H_*(BG;\mathbb{Z})$ and show it is reduced (no nilpotents).
- $H_*(BG;\mathbb{Z})$ is fin. gen. in each degree.
- $\implies$ We can assemble this computation from

$$H_*(BG;\mathbb{Z}[1/2]) \text{ and } H_*(BG;\mathbb{Z}_2)$$

$H_*(BString; \mathbb{Z}[1/2])$

- Serre SS $\implies$

$$H_*(BString; \mathbb{Q}) \hookrightarrow H_*(BSpin; \mathbb{Q})$$

$$H_*(BString; \mathbb{Q}) \hookrightarrow H_*(BSpin; \mathbb{Q}) \hookrightarrow H_*(BSO; \mathbb{Q})$$

  Moreover these are all polynomial algebras, so they are reduced.

- Have retraction

$$KO[1/2] \xrightarrow{\otimes \mathbb{C}} KU[1/2] \xrightarrow{U} KO[1/2].$$

- $\implies$ a retraction

$$H_*(BO\langle k\rangle; \mathbb{Z}[1/2]) \to H_*(BU\langle k\rangle; \mathbb{Z}[1/2]) \to H_*(BO\langle k\rangle; \mathbb{Z}[1/2]).$$

- Work of Hovey-Ravenel $\implies$ for $k \leq 8$, $H_*(BU\langle k\rangle; \mathbb{Z}[1/2])$ is torsion free.

$H_*(BString;\mathbb{Z}_2)$

- Stong's calculations $\implies$

Stong's calculations $\implies$

$$H_*(BO;\mathbb{F}_2) \hookleftarrow H_*(BSO;\mathbb{F}_2)$$

$$\begin{array}{ccc} & & H_*(BSpin;\mathbb{F}_2) \\ & & \big\downarrow \\ H_*(BO;\mathbb{F}_2) & \hookleftarrow & H_*(BSO;\mathbb{F}_2) \end{array}$$

$H_*(BString;\mathbb{Z})$

Proposition

$H_*(BString;\mathbb{Z})$, $H_*(BSpin;\mathbb{Z})$, $H_*(BSO;\mathbb{Z})$ have only simple 2-torsion.

$$H_*(BSO;\mathbb{Z}) \hookrightarrow H_*(BSO;\mathbb{F}_2) \times H_*(BSO;\mathbb{Q})$$

$$H_*(BSpin;\mathbb{Z}) \hookrightarrow H_*(BSpin;\mathbb{F}_2) \times H_*(BSpin;\mathbb{Q})$$

$\pi_* MString$

Lemma (Mathew-Naumann-N.)

To summarize there is a commutative diagram:

$$\begin{array}{ccccc}
\pi_* MString & \longrightarrow & H_*(MString;\mathbb{Z}) & \hookrightarrow & H_*(MString;\mathbb{F}_2)\times H_*(MString;\mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_* MSpin & \longrightarrow & H_*(MSpin;\mathbb{Z}) & \hookrightarrow & H_*(MSpin;\mathbb{F}_2)\times H_*(MSpin;\mathbb{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_* MSO & \hookrightarrow & H_*(MSO;\mathbb{Z}) & \hookrightarrow & H_*(MSO;\mathbb{F}_2)\times H_*(MSO;\mathbb{Q})
\end{array}$$

Theorem (Mathew-Naumann-N.)

This implies

$$\begin{aligned}
\ker(\pi_* MString \to \pi_* MSO) &= \ker(\pi_* MString \to H_*(MString;\mathbb{Z})) \\
&= \mathrm{Nil}(\pi_* MString).
\end{aligned}$$

and the result follows.

Further applications

- Have similar nilpotence results for $Spin$-manifolds, $U\langle 6\rangle$-manifolds, and SU-manifolds.
- Can deduce differentials in the Adams SS for connective E_∞-ring spectra.
- Can show ring spectra such as $BP/(p^3v_n^2), ku/(30\beta^5)$, and $tmf/4$ *do not* admit E_∞ ring structures.
- Can show 'half' of Quillen's F-isomorphism theorem for Lubin-Tate theories.
- We have since generalized this, with an independent argument, to general complex oriented cohomology theories for finite p-groups.

End

Thank you for your attention!