# On a nilpotence conjecture of J.P. May

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#### May's conjecture

#### Conjecture (May - 1986)

Suppose that R is a ring spectrum with power operations, i.e., an  $H_\infty$ -ring spectrum, and

 $x \in \ker(\pi_* R \to H_*(R;\mathbb{Z})).$ 

Then x is nilpotent., i.e.,  $x^n = 0$  for  $n \gg 0$ .

• When  $R = S \implies$  Nishida's nilpotence theorem.

• In contrast to the nilpotence theorem (Devinatz-Hopkins-Smith): We do not need to know about  $MU_*R$ , but now require R to be  $H_{\infty}$ .

## $E_{\infty}/H_{\infty}$ -rings

• An  $E_{\infty}$ -ring spectrum R has extended power maps:

$$\mu_n: E\Sigma_{n+} \wedge_{\Sigma_n} R^n \to R$$

- The  $\{\mu_n\}_{n\geq 0}$  are required to fit into some commutating diagrams.
- All of these diagrams are in Spectra.
- Analogous definition in ho(Spectra) yields  $H_{\infty}$ -ring spectra.
- Obtain forgetful functor:

$$U: ho(E_{\infty} - \text{Spectra}) \rightarrow H_{\infty} - \text{Spectra}.$$

• 
$$R$$
 is  $E_{\infty} \simeq \text{comm. } S$ -algebra  $\Longrightarrow R$  is  $H_{\infty}$ .

## Examples of $E_{\infty}/H_{\infty}$ -rings

Let X be a simply connected space of finite type.

k	Spaces		$H_{\infty}\mathrm{Mod}_{Hk} \subset k_* \mathrm{v.s.}$
Q	4	$H\mathbb{Q}^X \simeq C^*(X;\mathbb{Q})$	$H^*(X;\mathbb{Q})$
$\overline{\mathbb{F}}_p$	$X_p$	$H\overline{\mathbb{F}}_p^X \simeq C^*(X;\overline{\mathbb{F}}_p)$	$H^*(X;\overline{\mathbb{F}}_p)$

- Sullivan theory says we can recover  $X_{\mathbb{Q}}$  from the  $E_{\infty}$  algebra  $H\mathbb{Q}^X$ .
- Only get  $H^*(X; \mathbb{Q}) \in \mathbb{Q}$ -CAlg.
- Mandell's theory says we can recover  $X_p$  from the  $E_\infty\text{-algebra}$   $H\overline{\mathbb{F}}_p^X$  .
- Only get  $H^*(X; \overline{\mathbb{F}}_p)$  as an unstable algebra over Steenrod algebra.

#### Main result

#### Theorem (Mathew-Naumann-N.)

Suppose that R is an  $H_{\infty}$ -ring spectrum. Suppose that  $x \in \pi_*R$  has nilpotent image in  $H_*(R;\mathbb{Q})$  and  $H_*(R;\mathbb{F}_p)$  for  $\forall$  primes p. Then x is nilpotent.

#### Corollary

May's conjecture: Suppose that R is an  $H_\infty$ -ring spectrum and

 $x \in \ker(\pi_* R \to H_*(R;\mathbb{Z})).$ 

Then x is nilpotent.

#### **Proof reduction**

#### Theorem (Hopkins-Smith)

Suppose that R is a ring spectrum and that  $x \in \pi_*R$  has nilpotent image in

 $\bigcirc H_*(R;\mathbb{Q})$ 

- ②  $H_*(R; \mathbb{F}_p)$  for ∀ primes p
- **③** and  $K(n)_*(R)$  for  $\forall$  primes p and n > 0.

Then x is nilpotent.

#### Theorem (Mathew-Naumann-N.)

If R is  $H_{\infty}$  then condition (1) imply condition (3) and therefore (1) and (2) imply the main theorem.

## Proof of thm

- Since x has nilpotent image H<sub>\*</sub>(R; Q) ≅ π<sub>\*</sub>R ⊗ Q, ∃m s.t. x<sup>m</sup> is torsion.
- $y := x^m$  is nilpotent  $\iff x$  is nilpotent.
- WTS that our torsion y has nilpotent image in  $K(n)_*R$ .
- Hurewicz map factors

$$\pi_*R \to \pi_*L_{K(n)}(E_n \wedge R) \to K(n)_*(R).$$

•  $\operatorname{Im}(y) \in \pi_* L_{K(n)}(E_n \wedge R)$  is nilpotent  $\implies \operatorname{Im}(y) \in K(n)_*(R)$  is nilpotent  $\implies$  the main theorem.

#### Proof of thm II

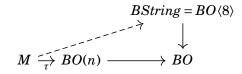
- $\pi_*L_{K(n)}(E_n \wedge R)$  is p-local.
- Assume  $y \in \pi_* L_{K(n)}(E_n \wedge R)$  is  $p^m$ -torsion.
- We then show that  $y^{p+1} \in \pi_* L_{K(n)}(E_n \wedge R)$  is  $p^{m-1}$ -torsion.
- So  $y^{(p+1)^m} = 0 \in \pi_* L_{K(n)}(E_n \wedge R)$  and the theorem follows.
- Argument uses analogues of  $\psi/\theta$ -operations and the formula:

$$y^p = \psi(y) + p\theta(y).$$

• This depends on work of Rezk and Strickland.

## String manifolds

Recall that a String structure on a smooth manifold M is



- Similar definitions for Spin-manifolds, SO-manifolds, ...
- $\rightsquigarrow E_{\infty}$  Thom spectra  $MString, MSO, \ldots$
- $\pi_*MString \cong \Omega^*_{String}, \pi_*MSpin \cong \Omega^*_{Spin} \dots$
- Forgetful maps:

$$MString \rightarrow MSpin \rightarrow MSO \rightarrow MO$$

## String bordism

#### Theorem (Mathew-Naumann-N.)

For a String manifold M. The following are equivalent:

- For  $n \gg 0$ ,  $M^{\times n}$  bounds a String manifold.
- 2 M bounds an oriented manifold.
- **3** The Stiefel-Whitney and Pontryagin numbers of M vanish.
  - (2)  $\iff$  (3) is classical.
  - Claim is equivalent to

Nil( $\pi_*MString$ ) = ker( $\pi_*MString \rightarrow \pi_*MSO$ ).

• We will show these are both are equal to

$$\ker(\pi_*MString \rightarrow H_*(MString;\mathbb{Z})).$$

#### $Nil(\pi_*MString)$

• May's conjecture  $\implies$ 

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\ker(\pi_*MString \to H_*(MString;\mathbb{Z})) \subseteq \operatorname{Nil}(\pi_*MString).
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Always have

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\operatorname{Im}(\operatorname{Nil}(\pi_*MString)) \subseteq \operatorname{Nil}(H_*(MString;\mathbb{Z})).
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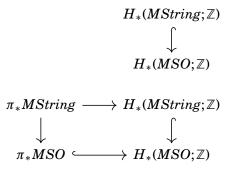
• We will show

 $\operatorname{Nil}(H_*(MString;\mathbb{Z})) \subseteq \operatorname{Nil}(H_*(MSO;\mathbb{Z})) = 0.$ 

 $\implies \ker(\pi_*MString \rightarrow H_*(MString;\mathbb{Z})) = \operatorname{Nil}(\pi_*MString).$ 

## $H_*(MString;\mathbb{Z})$

We will show:



 $\implies \ker(\pi_*MString \to \pi_*MSO) = \ker(\pi_*MString \to H_*(MString;\mathbb{Z})).$ 

So want to analyze  $H_*(MString;\mathbb{Z})$ .

#### Thom isomorphisms

- Have multiplicative isos  $H_*(MG;k) \cong H_*(BG;k)$  for G = SO, Spin, String.
- So we need to analyze  $H_*(BG;\mathbb{Z})$  and show it is reduced (no nilpotents).
- $H_*(BG;\mathbb{Z})$  is fin. gen. in each degree.
- ullet  $\Longrightarrow$  We can assemble this computation from

 $H_*(BG;\mathbb{Z}[1/2])$  and  $H_*(BG;\mathbb{Z}_2)$ 

## $H_*(BString; \mathbb{Z}[1/2])$

 $\bullet \ \, {\sf Serre} \ \, {\sf SS} \ \Longrightarrow \ \,$ 

 $H_*(BString; \mathbb{Q}) \hookrightarrow H_*(BSpin; \mathbb{Q})$ 

 $H_*(BString; \mathbb{Q}) \longrightarrow H_*(BSpin; \mathbb{Q}) \longrightarrow H_*(BSO; \mathbb{Q})$ Moreover these are all polynomial algebras, so they are reduced

Have retraction

$$KO[1/2] \xrightarrow{\otimes \mathbb{C}} KU[1/2] \xrightarrow{U} KO[1/2].$$

•  $\implies$  a retraction

 $H_*(BO\langle k\rangle;\mathbb{Z}[1/2]) \to H_*(BU\langle k\rangle;\mathbb{Z}[1/2]) \to H_*(BO\langle k\rangle;\mathbb{Z}[1/2]).$ 

• Work of Hovey-Ravenel  $\implies$  for  $k \le 8$ ,  $H_*(BU\langle k \rangle; \mathbb{Z}[1/2])$  is torsion free

## $H_*(\overline{BString};\mathbb{Z}_2)$

• Stong's calculations  $\implies$ 

Stong's calculations  $\implies$ 

$$H_*(BO;\mathbb{F}_2) \longleftrightarrow H_*(BSO;\mathbb{F}_2)$$
 $H_*(BSpin;\mathbb{F}_2)$ 
 $\downarrow$ 
 $H_*(BO;\mathbb{F}_2) \longleftrightarrow H_*(BSO;\mathbb{F}_2)$ 

## $H_*(BString;\mathbb{Z})$

#### Proposition

# $H_*(BString;\mathbb{Z}), H_*(BSpin;\mathbb{Z}), H_*(BSO;\mathbb{Z})$ have only simple 2-torsion.

#### $H_*(BSO;\mathbb{Z}) \longleftrightarrow H_*(BSO;\mathbb{F}_2) \times H_*(BSO;\mathbb{Q})$

$$H_*(BSpin;\mathbb{Z}) \longrightarrow H_*(BSpin;\mathbb{F}_2) \times H_*(BSpin;\mathbb{Q})$$

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### $\pi_*MString$

#### Lemma (Mathew-Naumann-N.)

To summarize there is a commutative diagram:

#### Theorem (Mathew-Naumann-N.)

This implies

$$\begin{aligned} \ker(\pi_*MString \to \pi_*MSO) &= \ker(\pi_*MString \to H_*(MString;\mathbb{Z})) \\ &= \operatorname{Nil}(\pi_*MString). \end{aligned}$$

and the result follows.

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### Further applications

- Have similar nilpotence results for Spin-manifolds,  $U\langle 6 \rangle$ -manifolds, and SU-manifolds.
- Can deduce differentials in the Adams SS for connective  $E_{\infty}$ -ring spectra.
- Can show ring spectra such as  $BP/(p^3v_n^2), ku/(30\beta^5)$ , and tmf/4 do not admit  $E_\infty$  ring structures.
- Can show 'half' of Quillen's *F*-isomorphism theorem for Lubin-Tate theories.
- We have since generalized this, with an independent argument, to general complex oriented cohomology theories for finite *p*-groups.



# Thank you for your attention!

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