On a nilpotence conjecture of J.P. May

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If R is H_∞ then condition (1) *imply condition* (3) *and therefore* (1) *and* (2) *imply the main theorem.*

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Thank you for your attention!