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ON AND AROUND SOME CONJECTURES OF AUSONI & ROGNES

ALGEBRAIC K-THEORY

$$\mathsf{K}(-)\colon \mathsf{Ring} \to \mathsf{Sp}_{\geq 0}$$

restricts to

$$\mathsf{K}(-)\colon \mathsf{CRing} \to \mathsf{CAlg}(\mathsf{Sp}_{\geq 0})$$

$$\mathsf{K}_0(\mathsf{R}) = \pi_0 \mathsf{K}(\mathsf{R}) = \mathbb{Z}\{\mathsf{Iso}(\mathsf{Mod}_{f.g.\mathsf{proj}}(\mathsf{R}))\}/\sim$$

Given: $0 \to \mathsf{A} \to \mathsf{B} \to \mathsf{C} \to 0$ $\qquad$ $[\mathsf{B}] \sim [\mathsf{C}] + [\mathsf{A}]$

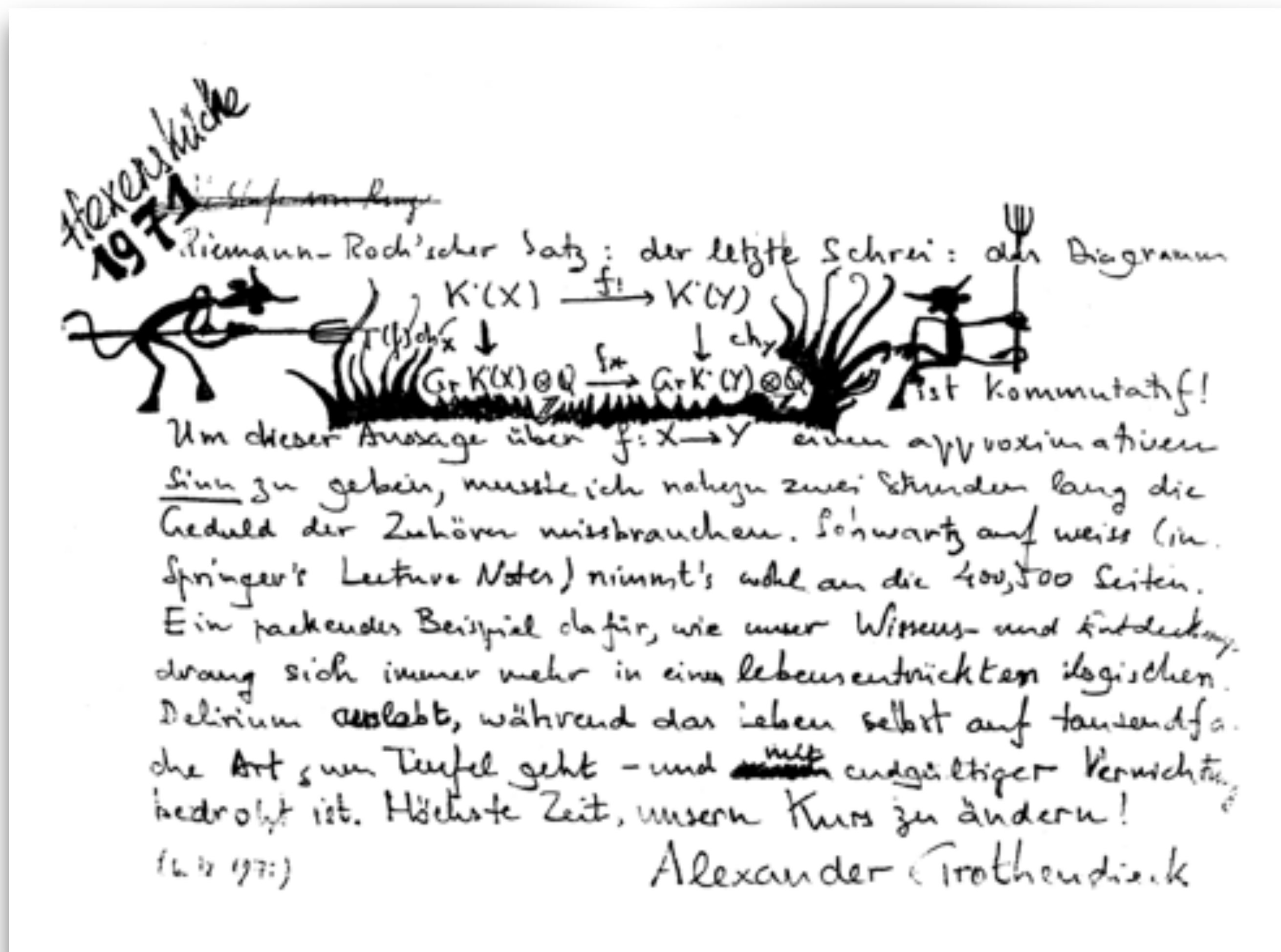
(K is also defined on exact cats, schemes $\mathsf{Cat}_\infty^{\mathsf{ex}}$, and Waldhausen cats)

YOU ARE SUPPOSED TO CARE ABOUT K-THEORY

$\mathrm{Pic}(R)$ $\mathrm{Br}(R)$ $\mathrm{CH}_i(R)$

All connect to $K_*(R)$

Grothendieck-Riemann-Roch Theorem:



Hexenküche 1971

Riemann-Roch'scher Satz: der letzte Schrei: das Diagramm

$$\begin{array}{ccc} K^\cdot(X) & \xrightarrow{f_!} & K^\cdot(Y) \\ \downarrow{\scriptstyle T(f)\,ch_X} & & \downarrow{\scriptstyle ch_Y} \\ Gr\,K^\cdot(X)\otimes\mathbb{Q} & \xrightarrow{f_*} & Gr\,K^\cdot(Y)\otimes\mathbb{Q} \end{array}$$

ist kommutativ!

Um dieser Aussage über $f: X \to Y$ einen approximativen Sinn zu geben, musste ich nahezu zwei Stunden lang die Geduld der Zuhörer missbrauchen. Schwarz auf weiss (in Springer's Lecture Notes) nimmt's wohl an die 400, 500 Seiten. Ein packendes Beispiel dafür, wie unser Wissens- und Entdeckungsdrang sich immer mehr in einem lebensentrückten logischen Delirium auslebt, während das Leben selbst auf tausendfache Art zum Teufel geht – und mit endgültiger Vernichtung bedroht ist. Höchste Zeit, unseren Kurs zu ändern!

Alexander Grothendieck

LICHTENBAUM–QUILLEN TYPE CONJECTURES

QUILLEN (1974 ICM)

`One might hope to have a spectral sequence, analogous to the Atiyah-Hirzebruch spectral sequence of topological K-theory:

$$E_2^{s,t} = H^s_{\text{ét}}(A; \mathbb{Z}_\ell(t/2)) \Rightarrow \pi_{t-s} K(A)_\ell$$

converging for t-s>dim(A)+1.'

THM. THOMASON (1985) THOMASON-TROBAUGH (1990)

Suppose A noetherian of finite Krull dimension, $\ell \in GL_1(A)$, and...

Then:

$$\exists E_2^{s,t} = H^s_{\text{ét}}(A; \mathbb{Z}_\ell(t/2)) \Rightarrow \pi_{t-s} L_{K(1)} K(A).$$

K(1) is the mod ℓ Morava K-theory.

LQ-TYPE CONJECTURES CONTINUED

LQ Conjecture remix:

$K(A)_\ell \to L_{K(1)}K(A)$ has coconnective fiber.

$\implies$ **THM. MITCHELL (1990)**

$L_{K(n)}K(A) = 0, \forall n \geq 2.$

$L_{K(n)}A = 0, \forall n \geq 1$ and $L_{K(1)}K(A)$ is usually not zero.

$\implies$ $K(-)$ raises chromatic complexity of discrete rings.

Red-shift!

BRAVE NEW RINGS

Waldhausen predicted K-theory extends:

$$K(-)\colon \mathrm{Ring} \subset \mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}_{\geq 0}$$

$$K(-)\colon \mathrm{CRing} \subset \mathrm{CAlg}(\mathrm{Sp}) \to \mathrm{CAlg}(\mathrm{Sp}_{\geq 0})$$

Moreover $S_{(p)} \xrightarrow{\simeq} \mathrm{holim}_n L_n S$

$\implies$

THM. MCCLURE-STAFFELT (1993)

$$K(S_{(p)}) = A(*;p) \xrightarrow{\simeq} \mathrm{holim}_n K(L_n S)$$

So K-theory of spaces can be studied via the chromatic filtration.

GALOIS DESCENT

Thomason's descent result reduces to Galois descent:

THM. THOMASON (1985)

$A \to B$ a finite G-Galois extension of fields satisfying ...

Then the induced map:

$$K(A) \to K(B)^{hG}$$

is an equivalence after $K(1)$-localization.

ROGNES-GALOIS EXTENSIONS

DEFN. ROGNES (2005)

Let $A \in \mathrm{CAlg}(\mathrm{Sp})$,

$B \in \mathrm{CAlg}(\mathrm{Mod}_A)$ equipped with a G-action.

Then $A \to B$ is a G-Galois extension if:

$$A \xrightarrow{\simeq} B^{hG}$$

$$B \wedge_A B \xrightarrow{\simeq} \prod_G B$$

The $K(n)$-local analogue is a $K(n)$-local G-Galois extension.

AUSONI-ROGNES GALOIS DESCENT CONJECTURE

Let $F(n+1)$ be a type $(n+1)$-finite ℓ-local spectrum with v_{n+1}-element v.

$$\text{Set } T := T(n+1) = F(n+1)[v^{-1}].$$

CONJ. AUSONI-ROGNES (2008)

CONJECTURE 4.2. *Let $A \to B$ be a $K(n)$-local G-Galois extension. Then there is a homotopy equivalence*

$$T \wedge K(A) \to T \wedge \left(K(B)\right)^{hG}.$$

Note $T(n+1)$-equivalence $\implies$ $K(n+1)$-equivalence.

AUSONI-ROGNES LQ-TYPE CONJECTURE

Let $V = F(n+1)$.

CONJ. AUSONI-ROGNES (2008)

CONJECTURE 4.3. *Let B be a suitably finite $K(n)$-local commutative $\mathbf{S}$-algebra (for example $L_{K(n)}\mathbf{S} \to B$ could be a G-Galois extension). Then the map $V \wedge K(B) \to T \wedge K(B)$ induces an isomorphism on homotopy groups in sufficiently high degrees.*

$$\Longrightarrow T(m)_* K(B) = K(m)_* K(B) = 0, \forall m > n+1.$$

$\Longrightarrow$ K-theory increases the bound on the chromatic complexity.

EVIDENCE FOR CONJECTURES

If $A \to B$ is a G-Galois extension of characteristic 0 fields,

and A satisfies Thomason's conditions,

the AR-Galois-descent conjecture holds by Thomason's theorem.

For the C_{p-1}-Galois extension $L_p \to KU_p$,

the AR-Galois-descent conjecture holds by work of Ausoni-Rognes.

EVIDENCE FOR CONJECTURES

When B is a field of char 0 satisfying Thomason's conditions,

then the Rost-Voevodsky proof of the Bloch-Kato conjecture

$\overset{\text{Levine}}{\Longrightarrow}$ the LQ=ARLQ-conjecture holds for B.

When $p \geq 5$, B $\in \{\mathrm{L}_p, \mathrm{KU}_p, l_p, ku_p\}$,

then, by calculations of Ausoni-Rognes,

the ARLQ-conjecture holds for B.

SOLUTIONS TO THE AR-GALOIS DESCENT CONJECTURE

THM. CLAUSEN-MATHEW-NAUMANN-N (2016)

Let $A \to B$ one of the following finite G-Galois extensions:

Any G-Galois extension of fields (no Thomason hypotheses)

$KO \to KU$ $\quad$ $E_n^{hG} \to E_n$, for $G \subset \mathbb{G}_n$ (Meier-Naumann-N)

Any G-Galois extension of $TMF[\frac{1}{n}]$

Any G-Galois extension of $Tmf_0(n)$

Then the induced maps:

$$L_T K(A) \to L_T(K(B)^{hG}) \to (L_T K(B))^{hG}$$

are equivalences for *any* periodic localization L_T.

(e.g., $L_{T(n)}, L_{K(n)}, L_n^f, L_n$)

FURTHER RESULTS ON DESCENT

With $A \to B$ and L_T as above:

There is an $N \geq 2$, such that associated HFPSS

$$E_2^{s,t} = H^s(G; \pi_t L_T K(B)) \Rightarrow \pi_{t-s} L_T K(A)$$

collapses at E_N with a horizontal vanishing line.

There are analogous (non-Galois) descent results when $A \to B$ is:

A faithfully flat, finite map, such that $\pi_* B$ is a projective $\pi_* A$-module.

$$ko \to ku \qquad\qquad tmf[\tfrac{1}{3}] \to tmf_1(3)$$

The same statements hold if we replace K with
K^B, THH, or TC.

AROUND THE ARLQ CONJECTURE

We also give a new proof of Mitchell's theorem:

Let $R \in \mathrm{Alg}(\mathrm{Mod}(\mathbb{Z}))$. Then

$$K(n)_* K(R) = 0, \forall n \geq 2 \text{ and implicit primes } p.$$

The new method also proves:

$$K(n)_* K(KU) = 0, \forall n \geq 3 \text{ and implicit primes } p \in \{2, 3, 5\}.$$

Combining this with Ausoni-Rognes Thm. for $p \geq 5$, gives the conclusion for all primes.

Galois descent and localization gives the result for KO, ku, and ko.

EASY CASE

Thomason observed proving $\mathbb{Q}$-Galois descent for fields is easy:

Given $A \to B$ a G-Galois extension of fields,

there is a transfer map: $K_0(B) \to K_0(A)$ which is $\mathbb{Q}$-surjective.

$$[B] \mapsto |G| \cdot [A]$$

A transfer argument now shows:

$$K(A) \otimes \mathbb{Q} \xrightarrow{\simeq} (K(B)^{hG}) \otimes \mathbb{Q} \xrightarrow{\simeq} (K(B) \otimes \mathbb{Q})^{hG}$$

Moreover the equivalences imply $K_0(B) \otimes \mathbb{Q} \twoheadrightarrow K_0(A) \otimes \mathbb{Q}$.

So surjectivity is necessary and sufficient for the equivalences!

THE TRANSFER ARGUMENT

$$A \to B,\ G\text{-Galois} \xrightarrow{\text{Merling, Barwick et al}} K_G(B) \in \mathrm{CAlg}(\mathrm{Sp}_G)$$

$$K(A) = K_G(B)^G \qquad K(B) = K_G(B)^e$$

Have fiber sequence:

$$F = \mathrm{Hom}_{K_G(B)}\left(\widetilde{E}G \wedge K_G(B), K_G(B)\right)^G \to K(A) \to K(B)^{hG}$$

Want to show $L_n^f F = 0$

(Take $n = 0$ for Thomason's argument)

F is an $(\widetilde{E}G \wedge K_G(B))^G$-module

Suffices to show $L_n^f(\widetilde{E}G \wedge K_G(B))^G = 0$

THE TRANSFER ARGUMENT

Have fiber sequence:

and a map: $K(B) \to K(B)_{hG} \xrightarrow{\text{Ind}} K(A) \to R = (\widetilde{E}G \wedge K_G(B))^G$

Composite $K(B) \to K(A)$ is the transfer.

Want to show $L_n^f R = 0$

By assumption $K_0(B) \otimes \mathbb{Q} \overset{tr}{\twoheadrightarrow} K_0(A) \otimes \mathbb{Q}$

So $\pi_0 R \otimes \mathbb{Q} = 0 \iff R \otimes \mathbb{Q} = 0$ (Thomason's argument)

$\overset{\text{May Conj}}{\iff}$ $L_n^f R = 0, \forall n \geq 0$ and primes p.

QED

SUMMARY FOR GALOIS DESCENT

So if $A \to B$ is a G-Galois extension then

the induced maps:

$$L_T(K(A)) \to L_T(K(B)^{hG}) \to (L_T K(B))^{hG}$$

are equivalences for any periodic localization L_T

if and only if

$$K_0(B) \otimes \mathbb{Q} \twoheadrightarrow K_0(A) \otimes \mathbb{Q}$$

This is the explicit condition we check for our examples.

BOUNDED CHROMATIC COMPLEXITY THEOREM

THEOREM (CLAUSEN-MATHEW-NAUMANN-N)

Let $E \in \mathrm{CAlg}(\mathrm{Sp})$ and $G = C_p^{\times n}$. If

$$\mathrm{Ind}_{\mathcal{P}}^{G} : \bigoplus_{A \subset G, |A| = p^{n-1}} E^0(BA) \otimes \mathbb{Q} \to E^0(BG) \otimes \mathbb{Q}$$

is surjective then

$$K(n+k)_* E = 0 \text{ for all } k \geq 0$$

Proof: $K(n+k) \wedge E \simeq * \iff R = L_{K(n+k)}(E_{n+k} \wedge E) \simeq *$

$\overset{\text{May Conj}}{\iff} \pi_0 R \otimes \mathbb{Q} = 0$

$$R^*(BC_p^{\times m}) = \oplus_{p^{(n+k)m}} R^*(pt.)$$

Use $E \to R$ and implied bound on ranks to see $R^*(pt.) \otimes \mathbb{Q} = 0$

BOUNDED CHROMATIC COMPLEXITY THEOREM

The condition:

$$\mathrm{Ind}_{\mathcal{P}}^{G}: \bigoplus_{A\subset G, |A|=p^{n-1}} E^0(BA)\otimes\mathbb{Q} \twoheadrightarrow E^0(BG)\otimes\mathbb{Q}$$

is hard to directly check when $E = K(R)$, $R \in \mathrm{CAlg}(\mathrm{Sp})$.

Instead consider:

$K(R, G)$=K-theory of R-modules with a G-action which are non-equivariantly compact.

There are natural maps $K(R, H) \to F(BH_+, K(R))$.

Suffices to check our condition on $K_0(R, H) \otimes \mathbb{Q}$.

MITCHELL TYPE THEOREMS

THEOREM (CLAUSEN-MATHEW-NAUMANN-N)

$$\mathrm{Ind}_{\mathcal{P}}^{C_p^{\times 2}} : \bigoplus_{H \subsetneq C_p^{\times 2}} K_0(\mathbb{Z}, H) \otimes \mathbb{Q} \to K_0(\mathbb{Z}, C_p^{\times 2}) \otimes \mathbb{Q}$$

is surj. for all primes p.

$$\mathrm{Ind}_{\mathcal{P}}^{C_p^{\times 3}} : \bigoplus_{H \subsetneq C_p^{\times 3}} K_0(KU, H) \otimes \mathbb{Q} \to K_0(KU, C_p^{\times 3}) \otimes \mathbb{Q}$$

is surj. for $p \in \{2, 3, 5\}$.

COR:

$K(2+k)_* K(\mathbb{Z}) = 0$ and $K(3+k)_* K(KU) = 0$ $\forall k \geq 0$ at these primes.

MITCHELL'S THEOREM

Regular rep and p-times a non-trivial character define

two maps $C_p \to U(p)$

which then define a map $C_p \times C_p \to PU(p)$.

Obtain a $C_p \times C_p$-action on $\mathbb{C}P^{p-1}$ with proper isotropy.

$$\implies [H\mathbb{Z} \wedge \mathbb{C}P^{p-1}] \in \operatorname{Im} \operatorname{Ind}_{\mathcal{P}}^{C_p \times C_p} \subset K_0(\mathbb{Z}, C_p \times C_p)$$

Postnikov filtration $\implies [H\mathbb{Z} \wedge \mathbb{C}P^{p-1}] = p[H\mathbb{Z}]$

So $\operatorname{Ind}_{\mathcal{P}}^{C_p \times C_p}$ is $\mathbb{Q}$-surjective.

BOUNDING CHROMATIC COMPLEXITY OF K(KU)

THEOREM (A. BOREL 1960)

There is an embedding $C_p^{\times 3} \to E_8$ which is not contained in a torus

$$\iff \ p \in \{2, 3, 5\}.$$

Obtain action of $C_p^{\times 3}$ on E_8/T with proper isotropy.

$$\implies \ [KU \wedge E_8/T] \in \mathrm{Im}\ \mathrm{Ind}_{\mathcal{P}}^{C_p^{\times 3}} \subset K_0(KU, C_p^{\times 3})$$

THEOREM (MATHEW-NAUMANN-N 2015)

There is an equivariant equivalence:

$$KU \wedge E_8/T \simeq \bigvee_{696729600} KU$$

So $\mathrm{Ind}_{\mathcal{P}}^{C_p^{\times 3}}$ is $\mathbb{Q}$-surjective.

THANK YOU!