Equivariant cohomology of representation spheres and $\mathbf{Pic}(S_G)$-graded homotopy groups

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Gradings

Question

What should homology and cohomology groups be indexed over?

- Normally $H_*(X)$ and $H^*(X)$ is graded over $\mathbb{N}$.
- Generalized (co)homology (e.g., K-theory) is graded over $\mathbb{Z}$.
- Equivariant (co)homology is graded over $\mathbb{N}$ and sometimes $RO(G)$.

One way to think of gradings

- Cohomology theories $\leftrightarrow$ spectra.
- Suppose $X \in \mathcal{T}op_*$ '$\subset$' $\mathcal{S}pectra$, $M \in \mathcal{A}b\mathcal{G}roup$.
- Then $\exists HM$ in spectra, satisfying

$$[S^i, HM \wedge X] \cong H_i(X; M)$$

  (always reduced).
- $[S^i \wedge X, HM] \cong H^{-i}(X)$ ($\exists S^{-i}$).
- Similarly for generalized (co)homology.

Indexing by spheres

- $E_*(-)$, $E^*(-), * \in \mathbb{Z}$.
- $\mathbb{Z} \cong \{S^i\}_{i \in \mathbb{Z}}$.
- $i + j \in \mathbb{Z} \leftrightarrow S^i \wedge S^j \simeq S^{i+j}$.
- So $\mathbb{Z}$-grading $\leftrightarrow$ 'sphere-grading.'
- Sign conventions come from

$$\tau \colon S^i \wedge S^j \to S^j \wedge S^i$$

which has degree $(-1)^{ij}$.

Why spheres?

- They are small, so wedge axiom holds:

$$E_i(\bigvee_{j\in J} X_j) \cong \bigoplus_{j\in J} E_i(X_j).$$

- The existence of inverses gives suspension isomorphisms:

$$\begin{aligned} E_i(\Sigma X) &\cong E_i(S^1 \wedge X) \\ &\cong [S^i, E \wedge S^1 \wedge X] \\ &\cong [S^i \wedge S^{-1}, E \wedge S^1 \wedge X \wedge S^{-1}] \\ &\cong [S^{i-1}, E \wedge X] \\ &\cong E_{i-1}(X). \end{aligned}$$

Requirements for indices

1. Wedge axiom $\implies$ Indexing objects are 'small' (dualizable).
2. Suspension axiom $\implies$ Indexing objects are invertible.
3. The (abelian) group of such objects is called the Picard group.

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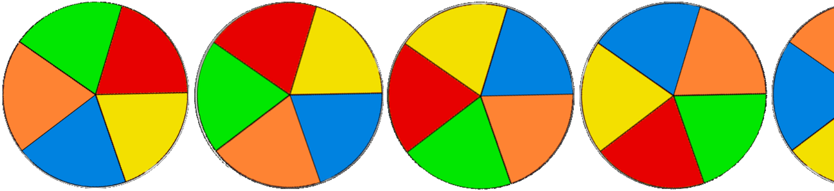
- Can define Picard group, $\mathrm{Pic}(\mathcal{C})$ for any symmetric monoidal category $\mathcal{C}$ (dualizable, w/ inverses).
- $\mathrm{Pic}(S) \cong \mathbb{Z}$, so nothing new here.
- Want 'more spheres.'

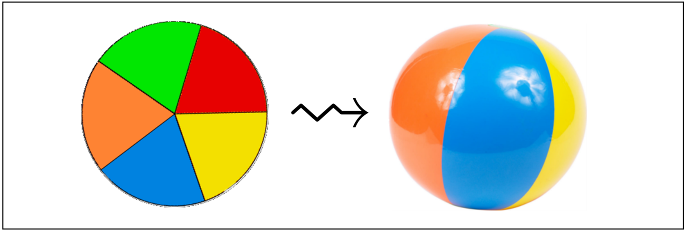
Representation spheres

- Let G be a finite group.
- Take an orthogonal G representation $V = \mathbb{R}^n \circlearrowleft G$.
- Here is a picture of the unit disc of a non-trivial representation of C_5.



G-CW structure

- To construct S^V, collapse the boundary $S(V)$, of the unit disc in V to a point.



- Construct a CW-decomposition on S^V, such that G takes cells to cells while never mapping a cell to itself in a non-trivial way.
- E.g., the color slices above.

$RO(G)$ vs. $\mathrm{Pic}(S_G)$

- Construction gives a morphism $RO(G) \to \mathrm{Pic}(S_G)$.
- Factors as
$$RO(G) \twoheadrightarrow JO(G) \hookrightarrow \mathrm{Pic}(S_G).$$
- $JO(G) := RO(G)/(\sim)$.
- $V \sim W \Leftrightarrow S^V \simeq S^W$.
- Let us find a toy case for computations.

Known results

Theorem (tom Dieck-Petrie)

$$\textit{Rank } JO(G) = \textit{Rank } \mathrm{Pic}(S_G) \Longleftrightarrow G \textit{ is nilpotent.}$$

Theorem (Kawakubo)

$$JO(G) \cong \mathrm{Pic}(S_G) \Longleftrightarrow G = C_n \textit{ or } D_{2\cdot 2^n}.$$

$\mathrm{Pic}(S_{C_n})$

Theorem (Kawakubo)

$$JO(S_{C_n}) \cong \mathrm{Pic}(S_{C_n}) \cong \bigoplus_{d|n} \left(\mathbb{Z} \oplus (\mathbb{Z}/d\mathbb{Z}^\times/\langle \pm 1\rangle)\right).$$

- Torsion free summands generated by a rotation of order d.
- Torsion summand generated by differences of such representations.

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- Let's compute the $\mathrm{Pic}(S_G)$-graded homotopy of something.
- HM?!
- Essentially need to compute the equivariant homology and cohomology of these invertible objects (e.g., representation spheres).
- How do we do this?
- What should M be equivariantly?

Reminder: Cellular homology

- Given a CW-complex X let

$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\}$$
$$\widetilde{C}_0(X) := \ker(C_0(X) \to C_0(*)).$$

- $H_*(X) \cong H_*\left[\cdots C_{i+1}(X) \xrightarrow{\partial} C_i(X) \xrightarrow{\partial} C_{i-1}(X) \cdots \xrightarrow{\partial} \widetilde{C}_0(X)\right]$
- $H^*(X)$ is calculated by taking the dual of this complex and then taking cohomology.

Bredon homology with coefficients in $\mathbb{Z}$

- Given a G-CW-complex X let
$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\} \circlearrowleft G.$$
- For a subgroup $K \leqslant G$, let $C_i^K(X) \subset C_i(X)$ be the subgroup of K-invariant chains.
- $H_*^K(X) \cong H_*\left[\cdots C_{i+1}^K(X) \xrightarrow{\partial} C_i^K(X) \xrightarrow{\partial} C_{i-1}^K(X) \cdots \xrightarrow{\partial} \widetilde{C}_0^K(X)\right].$
- When K is the trivial subgroup: $H_*^K(X) \cong H_*(X)$.
- For cohomology first take the invariants on the cochains.

Mackey functors

- Other coefficient systems?
- Want $H_i^{(-)}(X)$ to also be an acceptable coefficient system.
- These functors assign an abelian group to each subgroup of G and have induction, restriction, and action maps, satisfying some axioms.
- Such functors should form an abelian category.

Definition: Mackey functors

Definition

$\mathscr{O}(G)$ is the category of finite G-sets and G-morphisms.

Definition

A Mackey functor is a pair

$$M_* : \mathscr{O}(G) \to \mathcal{AbGroup}$$
$$M^* : \mathscr{O}(G)^{\mathrm{op}} \to \mathcal{AbGroup}$$

such that

- $M_*(X) = M^*(X)$
- $M^*(X \coprod Y) = M^*(X) \times M^*(Y)$
- M satisfies a double coset formula.

Definition explained

- Alternatively M assigns to each $H \leqslant G$ an abelian group $M(G/H)$.
- $M(G/H)$ has action of $W_G H = N_G H/H$.
- From $H \leqslant K$ we obtain a map $\pi : G/H \to G/K$ inducing a *restriction* maps

$$M(G/H) \xrightarrow{M_*\pi = \mathrm{Res}_H^K} M(G/K)$$

and a *transfer* map

$$M(G/K) \xrightarrow{M^*\pi = \mathrm{Ind}_H^K} M(G/H).$$

Example: G-Modules

- Given a G-module M we can construct a functor $M(-)$:
- $M(G/H) = \mathcal{Mod}_G(\mathbb{Z}[G/H], M) \cong M^H$.
- For $K \leqslant H$,

$$M(G/H) = M^H \xrightarrow{\mathrm{Res}_K^H} M(G/K) = M^K$$

induced by quotient map

$$q : \mathbb{Z}[G/K] \to \mathbb{Z}[G/H].$$

- We also have a transfer

$$M(G/K) = M^K \xrightarrow{\mathrm{Ind}_K^H} M(G/H) = M^H$$

induced by summing over the fibers of q.

Example: Burnside ring

Definition

Let $A(G)$ be the Grothendieck group of finite G-sets up to iso.

- $[X \coprod Y] = [X] + [Y]$
- $[X \times Y] = [X][Y]$

- $G/H \mapsto A(H)$ is a Mackey functor.
- Restriction of G-action to H-action defines Res_H^G.
- Crossing with G/H defines Ind_H^G.

$A(C_9)$

Note

- $X = \coprod_i G/H_i$ (Decomposition into orbits).
- Rank $A(G) = \#$ Conjugacy classes of subgroups of G.

Example (Multiplication Table)

$$A(C_9) = \mathbb{Z}\{[C_9/C_9],[C_9/C_3],[C_9]\}$$

	$[C_9/C_9]$	$[C_9/C_3]$	$[C_9]$
$[C_9/C_9]$	$[C_9/C_9]$	$[C_9/C_3]$	$[C_9]$
$[C_9/C_3]$	$[C_9/C_3]$	$3[C_9/C_3]$	$3[C_9]$
$[C_9]$	$[C_9]$	$3[C_9]$	$?9[C_9]$

$C_9 \times C_9$ is a free C_9 set with 81 elements.

Example: Burnside ring

- $A(H)$ is a ring such that A is a *commutative Green functor*.
- Res^H_K is a commutative ring map.
- $\mathrm{Ind}^H_K(a) \cdot b = \mathrm{Ind}^H_K(a \cdot \mathrm{Res}^H_K(b))$.
- Every Mackey functor is an A-module.
- Analogue of $\mathbb{Z}$ equivariantly.

Induced Mackey functors

Definition

Given a Mackey functor M let $M \otimes G/H$ be the Mackey functor defined by

$$(M \otimes G/H)(G/K) = M(G/H \times G/K)$$

Example

$$A \otimes G \cong \mathbb{Z}[G]$$

Bredon homology with coefficients in M

- Given a G-CW-complex X let

$$C_i(X) : M \otimes \{i\text{-cells of } X\} \circlearrowleft G.$$

- Chain complex of Mackey functors.

$$H_*^K(X) \cong H_*\left[\cdots C_{i+1}(X)(G/K) \xrightarrow{\partial} C_i(X)(G/K) \xrightarrow{\partial} C_{i-1}(X)(G/K) \cdots \xrightarrow{\partial} \widetilde{C}_0(X)(G/K)\right]$$

- When $M = \mathbb{Z}$ (trivial action) then we get previous definition.
- For cohomology one takes a dual complex, with $\mathrm{Ind}_H^G \leftrightarrow \mathrm{Res}_H^G$.

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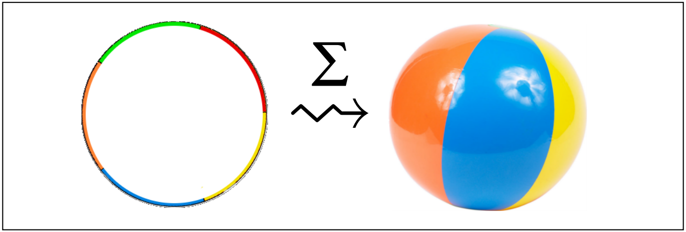
Method of computating $\pi_\star HA$

- Fix a finite group G and determine explicit models for all of the irreducible *real* representations of G.
- Construct an explicit G-CW decomposition on each irreducible representation sphere.
- Compute $H_*^G(S^V)$ and $H^*_G(S^V)$.
- Assemble the computations to compute $H_*^G(S^{V+W})$.

Homology

Consider ρ_n the rotation of order n $(n > 2)$



$$A \xleftarrow{\epsilon} A \otimes C_n \xleftarrow{g-1} A \otimes C_n$$

$H_*^e(S^{\rho_n};A)$

$$A \xleftarrow{\epsilon} A \otimes C_n \xleftarrow{g-1} A \otimes C_n$$

Evaluate at C_n/e:

$$\mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}^{|C_n|} \xleftarrow{[g]-[1]} \mathbb{Z}^{|C_n|}$$

$$H_0^e(S^{\rho_n}) \cong 0 \qquad H_1^e(S^{\rho_n}) \cong 0 \qquad H_2^e(S^{\rho_n}) \cong \mathbb{Z}$$

Get $H_*(S^2)$ as expected.

$H_*^{C_n}(S^{\rho_n};A)$

$$A \xleftarrow{\epsilon} A \otimes C_n \xleftarrow{g-1} A \otimes C_n$$

Evaluate at C_n/C_n:

$$A(C_n) \xleftarrow{\mathrm{Ind}_e^{C_n}} \mathbb{Z} \xleftarrow{0} \mathbb{Z}$$

$$H_0^{C_n}(S^{\rho_n}) \cong A(C_n)/[C_n] \qquad H_1^{C_n}(S^{\rho_n}) \cong 0 \qquad H_2^{C_n}(S^{\rho_n}) \cong \mathbb{Z}$$

$H^*_{C_n}(S^{\rho_n};A)$

$$A \xrightarrow{\Delta} A \otimes C_n \xrightarrow{g^{-1}-1} A \otimes C_n$$

Evaluate at C_n/C_n:

$$A(C_n) \xrightarrow{\mathrm{Res}_e^{C_n}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H^0_{C_n}(S^{\rho_n}) \cong \tilde{A}(C_n) = \ker(A(C_n) \xrightarrow{\mathrm{Res}_e^{C_n}} \mathbb{Z})$$
$$H^1_{C_n}(S^{\rho_n}) \cong 0$$
$$H^2_{C_n}(S^{\rho_n}) \cong \mathbb{Z}.$$

Assembling the computation

- Once we know $C_*(S^V)$ and $C_*(S^W)$ have

$$H_*(S^{V+W}) \cong H_*(C_*(S^W) \otimes C_*(S^V))$$

- Can filter bicomplex in two ways to get two spectral sequences.
- Alternatively these are AHSS's with coefficients in

$$E^2_{s,t} = H_s(S^V;\ H_t(S^W)) \implies H_{s+t}(S^{V+W}),$$
$$E^2_{s,t} = H_s(S^W;\ H_t(S^V)) \implies H_{s+t}(S^{V+W}).$$

Tricks for computation

1. (Reverse induction) If V is the pullback of a representation of G/H, then can pullback complex.
2. (Reciprocity) Use the formula

$$S^W \wedge \mathrm{Ind}_H^G S^i \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G(S^W) \wedge S^i\right)$$

   to simplify E_1.
3. (Functoriality) Use subgroup functoriality to determine the differentials and multiplicative relations.
4. (Competing computations) Decompose the representation in different ways.

$H_*^{C_n}(S^{2\rho_n})$

$$M \xleftarrow{\epsilon} M \otimes C_n \xleftarrow{g-1} M \otimes C_n$$

Evaluate at C_n/C_n with $M = H_*(S^{\rho_n})$:

$$H_*^{C_n}(S^{\rho_n}) \xleftarrow{\mathrm{Ind}_e^{C_n}} H_*^e(S^2) \xleftarrow{0} H_*^e(S^2)$$

$$\begin{aligned} H_0^{C_n}(S^{2\rho_n}) &\cong A(C_n)/[C_n] \\ H_{\mathrm{odd}}^{C_n}(S^{2\rho_n}) &\cong 0 \\ H_2^{C_n}(S^{2\rho_n}) &\cong \mathbb{Z}/(n) \\ H_4^{C_n}(S^{2\rho_n}) &\cong \mathbb{Z} \end{aligned}$$

$H_*^{C_n}(S^{m\rho_n};A)$

- One can compute these groups inductively and get a regular pattern.
- $H_0^{C_n}((S^{m\rho_n}) \cong A(C_n)/([C_n])$.
- $H_{\text{odd}}^{C_n}((S^{m\rho_n}) \cong 0$.
- For $0 < i < m$, $H_{2i}^{C_n}(S^{m\rho_n}) \cong \mathbb{Z}/(n)$.
- $H_{2m}^{C_n}(S^{m\rho_n}) \cong \mathbb{Z}$.
- Can also compute directly from a single chain complex.

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Multiplicative relations

- There are also external products

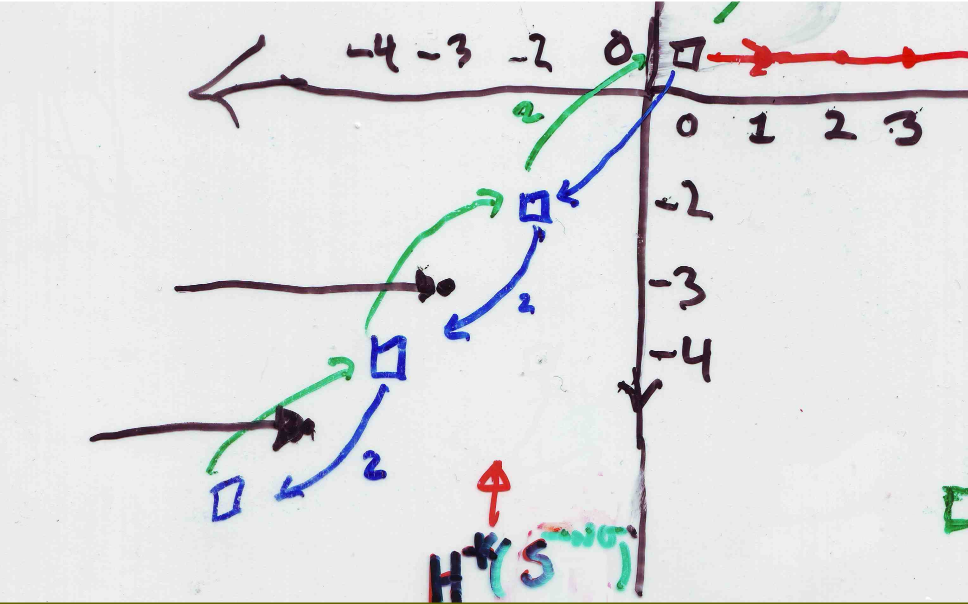
$$H_*^G(S^V) \otimes H_*^G(S^W) \to H_*^G(S^{V+W}).$$

- These relate the (co)homology groups of different representation spheres.
- Can deduce from spectral sequence and Green functor relations.

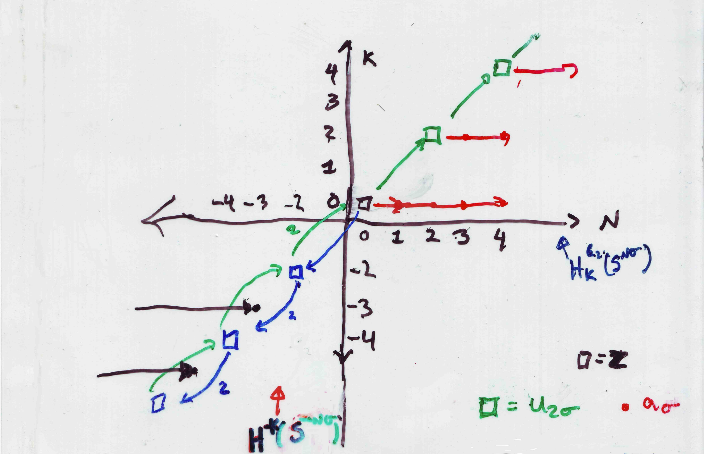
Some Relations

- $i_*(1) = a_V \in H_0^G(S^V)$ induced by inclusion of $S^0 \to S^V$.
- V orientable, $u_V \in H^G_{|V|}(S^V)$ (generates top class).
- V orientable, $v_V \in H_G^{|V|}(S^V)$ (generates top class).
- $a_V a_W = a_{V+W}$.
- $u_V u_W = u_{V+W}$.
- $v_{\rho_n} u_{\rho_n} = [C_n]$
- $v_{\rho_n} v_{\rho_n} = [C_n] v_{2\rho_n}$

Computations for $G = C_2$, $M = \mathbb{Z}$



Computations for $G = C_2$ $M = \mathbb{Z}$



$H_*^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A)$

Brace yourselves $C_*(S^{\rho_5}) \otimes A$

$$\begin{array}{cccc} & 0 & 1 & 2 \\ 0 & A & \xleftarrow{\epsilon} A \otimes C_5 & \xleftarrow{g-1} A \otimes C_5 \end{array}$$

$C^*(S^{\rho_5}) \otimes A \cong C_{-*}(S^{-\rho_5}) \otimes A$

$$\begin{tikzcd}[row sep=large] & 0 \\ 0 & A \arrow[d, "\Delta"] \\ -1 & A \otimes C_5 \arrow[d, "g^k - 1"] \\ -2 & A \otimes C_5 \end{tikzcd}$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

Compute cohomological ($\downarrow$) direction first.

$$
\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
 & 0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0
\end{array}
$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

Compute homological ($\leftarrow$) direction first.

	0	1	2
0	$A(C_5)/[C_5]$	0	$\mathbb{Z}$
-1	0	0	$\mathbb{Z}$
-2	0	0	$\mathbb{Z}$

	0	1	2
0	$A(C_5)/[C_5]$	0	$\mathbb{Z}$
			$?$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

Resolve differentials for first spectral sequence.

$$\begin{tikzcd}
 & 0 & 1 & 2 \\
0 & A(C_5)/[C_5] & 0 & \mathbb{Z} \arrow[d, "1"] \\
-1 & 0 & 0 & \mathbb{Z} \arrow[d, "0"] \\
-2 & 0 & 0 & \mathbb{Z}
\end{tikzcd}$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

Resolve differentials for second spectral sequence.

$$\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & \xleftarrow[1]{\quad\quad} \mathbb{Z} & \xleftarrow[0]{\quad\quad} \mathbb{Z}
\end{array}$$

$$\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0
\end{array}$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

- Our extension problem:

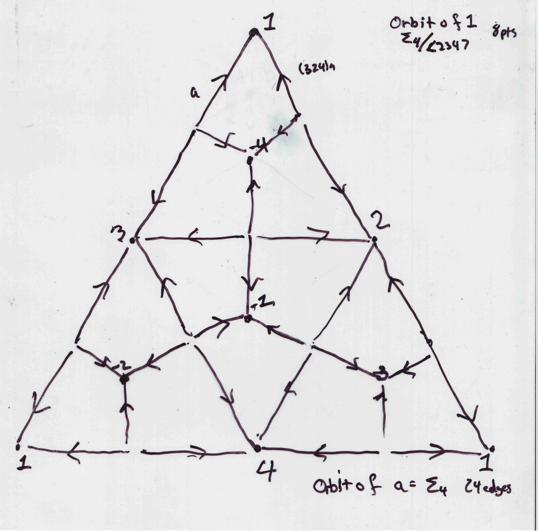
$$0 \to \tilde{A}(C_5) \to H_0^{C_5}(S^{\rho_5-\rho_5^{\otimes k}}) \to \mathbb{Z} \to 0$$

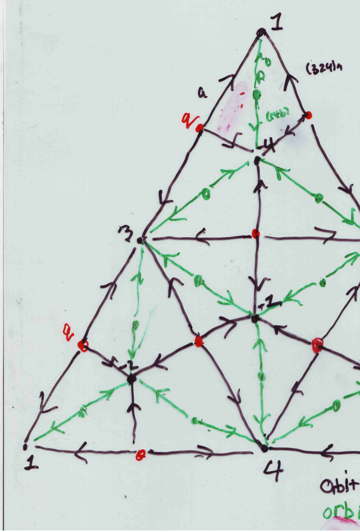
- Extension splits additively, but not as $A(C_5)$ modules.
- Use bicomplex to solve extension.
- If $k = \pm 1 \mod 5$ get $A(C_5) = H_0^{C_5}(S^0)$.
- If $k = \pm 2 \mod 5$ get a projective $A(C_5)$ module of rank one.

Summary

- We can also determine explicit models for the irreducible real representations of C_n, D_n, A_4, and S_4.
- We can compute the homology and cohomology of these representation spheres.
- Calculating $H_*^G(S^{V+W})$ is generally difficult due to complications in the spectral sequence.
- There is still plenty of other computations left to do.

Twisted tetrahedral representation of Σ_4





End