Equivariant cohomology of representation spheres and $\mathrm{Pic}(S_G)$-graded homotopy groups

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Gradings

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- Generalized (co)homology (e.g., K-theory) is graded over $\mathbb{Z}$.
- Equivariant (co)homology is graded over $\mathbb{N}$ and sometimes $RO(G)$.

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- Similarly for generalized (co)homology.

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  $$\tau \colon S^i \wedge S^j \to S^j \wedge S^i$$

  which has degree $(-1)^{ij}$.

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3. The (abelian) group of such objects is called the Picard group.

Päuschen

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- Want ‘more spheres.’

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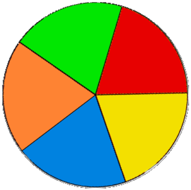

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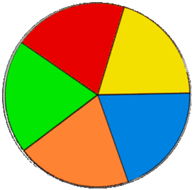

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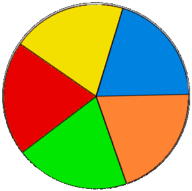

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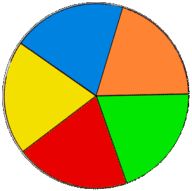

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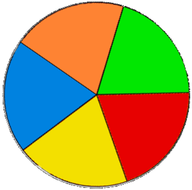

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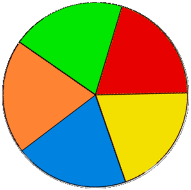

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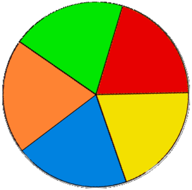

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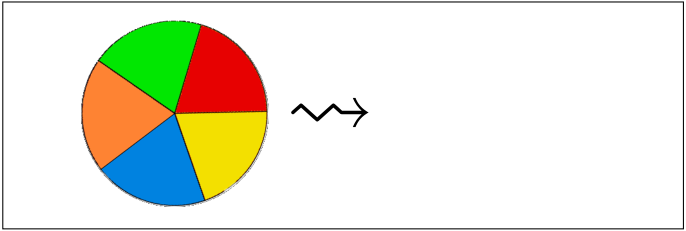

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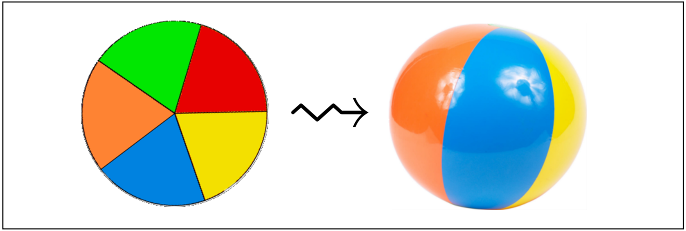

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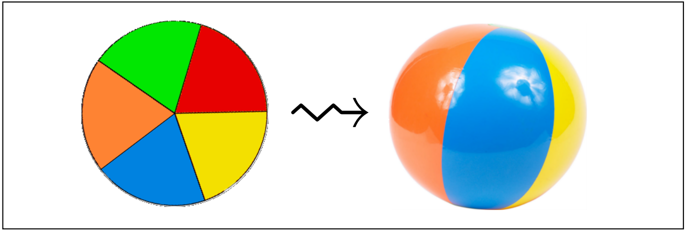

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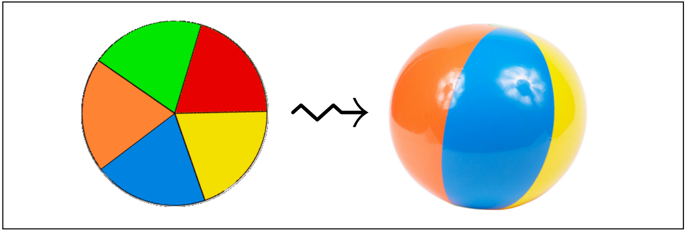

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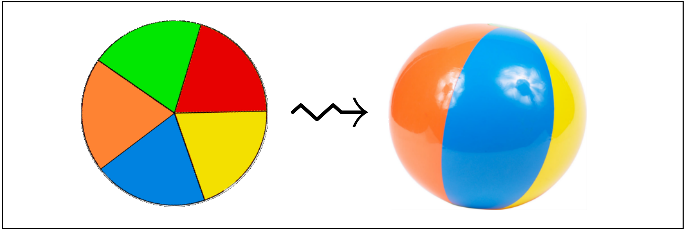

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- E.g., the color slices above.

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- Let us find a toy case where we can compute groups indexed over $\mathrm{Pic}(S_G)$.

Known results

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- Torsion summand generated by differences of such representations.

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- How do we do this?
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- $H^*(X)$ is calculated by taking the dual of this complex and then taking cohomology.

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- For cohomology first take the invariants on the cochains.

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- Such functors should form an abelian category.

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- M satisfies a double coset formula.

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$$A \otimes G \cong \mathbb{Z}[G]$$

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- For cohomology one takes a dual complex, with $\mathrm{Ind}_H^G \leftrightarrow \mathrm{Res}_H^G$.

Päuschen

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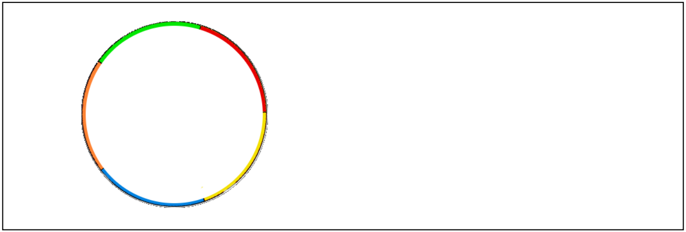

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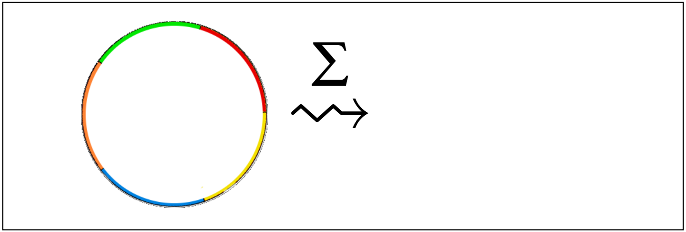

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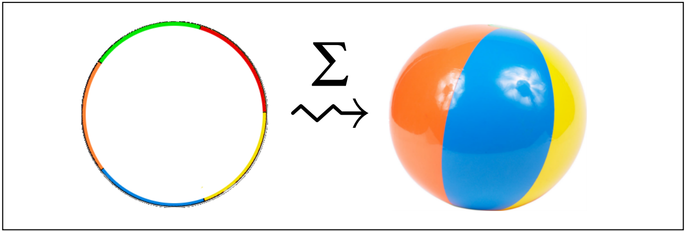

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$$E^2_{s,t} = H_s(S^V;\ H_t(S^W)) \Longrightarrow H_{s+t}(S^{V+W}),$$

Assembling the computation

- Once we know $C_*(S^V)$ and $C_*(S^W)$ have

$$H_*(S^{V+W}) \cong H_*(C_*(S^W) \otimes C_*(S^V))$$

- Can filter bicomplex in two ways to get two spectral sequences.
- Alternatively these are AHSS's with coefficients in

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3. (Functoriality) Use subgroup functoriality to determine the differentials and multiplicative relations.
4. (Competing computations) Decompose the representation in different ways.

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- Can also compute directly from a single chain complex.

Päuschen

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- Can deduce from spectral sequence and Green functor relations.

Some Relations

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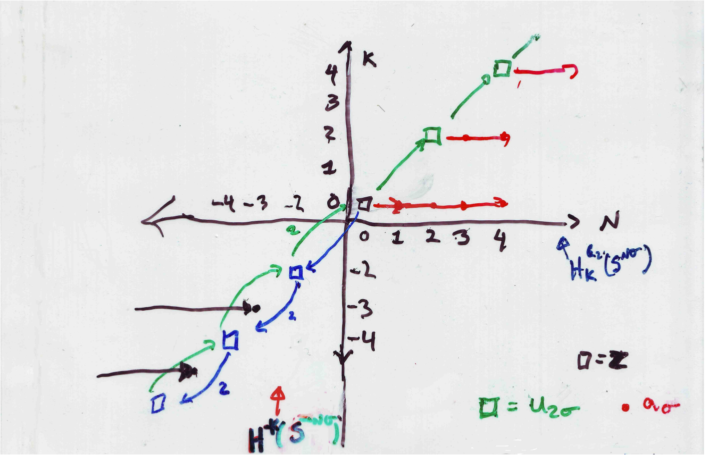
Computations for $G = C_2$, $M = \mathbb{Z}$



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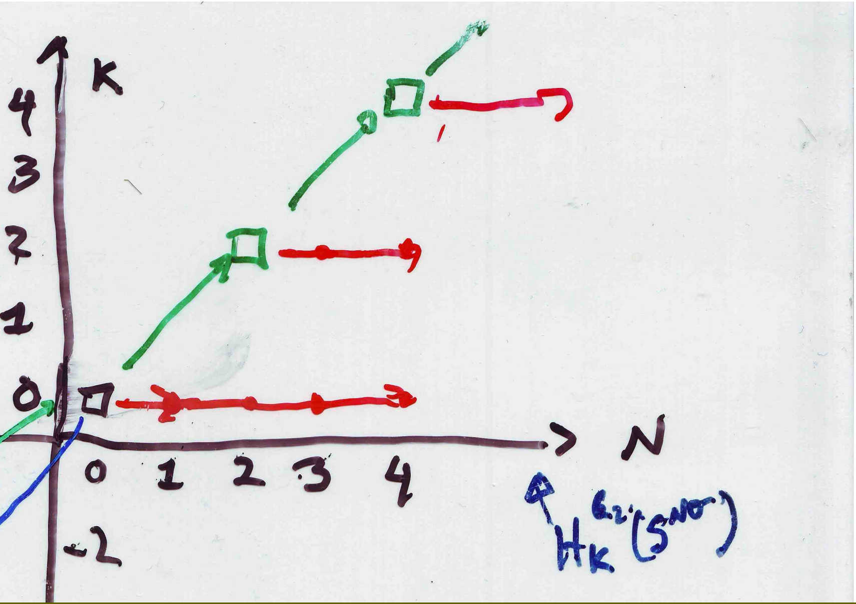

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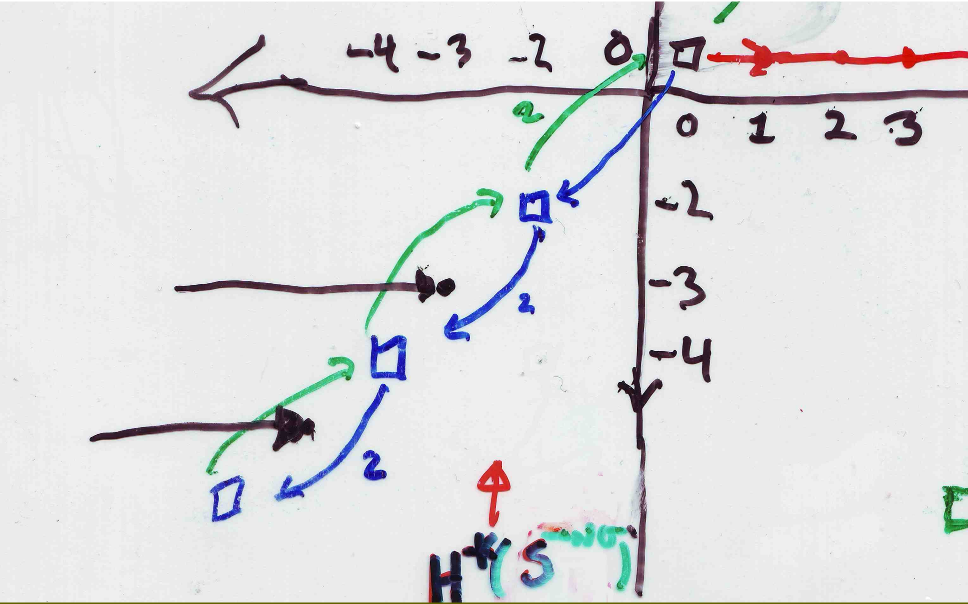

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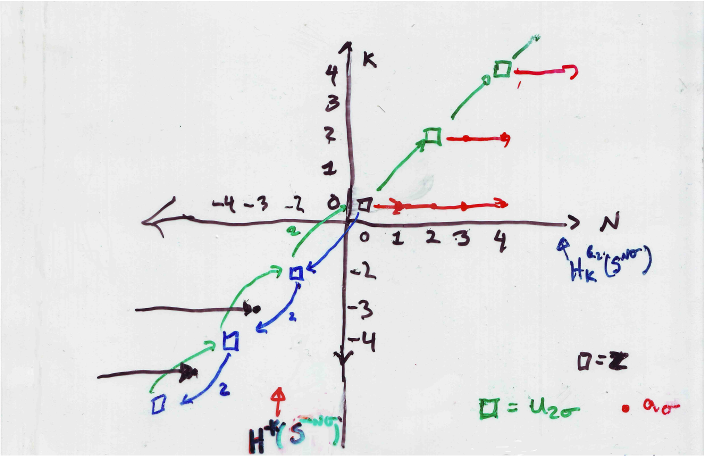

$H_*^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A)$

Brace yourselves

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

$C_*(S^{\rho_5})\otimes A$

$$
\begin{array}{cccccc}
 & 0 & & 1 & & 2 \\
0 & A & \xleftarrow{\epsilon} & A\otimes C_5 & \xleftarrow{g-1} & A\otimes C_5
\end{array}
$$

$H_*^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A)$

$$C^*(S^{\rho_5}) \otimes A \cong C_{-*}(S^{-\rho_5}) \otimes A$$

$$\begin{array}{cc} & 0 \\ 0 & A \\ & \downarrow \Delta \\ -1 & A \otimes C_5 \\ & \downarrow g^k - 1 \\ -2 & A \otimes C_5 \end{array}$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

$$\begin{tikzcd}
 & 0 & 1 & 2 \\
0 & A & A \otimes C_5 \arrow[l, "\epsilon"'] & A \otimes C_5 \arrow[l, "g-1"'] \\
-1 & A \otimes C_5 \arrow[from=u, "\Delta"] & A \otimes C_5 \otimes C_5 \arrow[l, "-[\epsilon]"'] \arrow[from=u, "{[\Delta]}"] & A \otimes C_5 \otimes C_5 \arrow[l, "{[1]-[g]}"'] \arrow[from=u, "{[\Delta]}"] \\
-2 & A \otimes C_5 \arrow[from=u, "g^k-1"] & A \otimes C_5 \otimes C_5 \arrow[l, "{[\epsilon]}"] \arrow[from=u, "{[g^k]-[1]}"] & A \otimes C_5 \otimes C_5 \arrow[l, "{[g]-[1]}"] \arrow[from=u, "{[g^k]-[1]}"]
\end{tikzcd}$$

$H_*^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A)$

$$
\begin{tikzcd}
 & 0 & 1 & 2 \\
0 & A \arrow[d, "\Delta"] & A \otimes C_5 \arrow[l, "\epsilon"'] \arrow[d, "{[\Delta]}"] & A \otimes C_5 \arrow[l, "g-1"'] \arrow[d, "{[\Delta]}"] \\
-1 & A \otimes C_5 \arrow[d, "g^k-1"] & A \otimes C_5 \otimes C_5 \arrow[l, "{-[\epsilon]}"'] \arrow[d, "{[g^k]-[1]}"] & A \otimes C_5 \otimes C_5 \arrow[l, "{[1]-[g]}"'] \arrow[d, "{[g^k]-[1]}"] \\
-2 & A \otimes C_5 & A \otimes C_5 \otimes C_5 \arrow[l, "{[\epsilon]}"] & A \otimes C_5 \otimes C_5 \arrow[l, "{[g]-[1]}"]
\end{tikzcd}
$$

Diagonal gives total degree.

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Compute cohomological ($\downarrow$) direction first.

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	0	1	2
0	$\tilde{A}(C_5)$	0	0
−1	0	0	0
−2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$H_*^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}; A)$

Compute cohomological (↓) direction first.

$$
\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & \tilde{A}(C_5) & 0 & 0 \\
-1 & 0 & 0 & 0 \\
-2 & \mathbb{Z} \xleftarrow[?]{} & \mathbb{Z} \xleftarrow[?]{} & \mathbb{Z}
\end{array}
$$

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0	$A(C_5)/[C_5]$	0	$\mathbb{Z}$
-1	0	0	$\mathbb{Z}$
-2	0	0	$\mathbb{Z}$

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 & 0 & 1 & 2 \\
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 & & & \downarrow ? \\
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$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

Resolve differentials for first spectral sequence.

$$\begin{array}{cccc}
 & 0 & 1 & 2 \\
0 & A(C_5)/[C_5] & 0 & \mathbb{Z} \\
 & & & \downarrow 1 \\
-1 & 0 & 0 & \mathbb{Z} \\
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Resolve differentials for second spectral sequence.

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We have an extension:

$$0 \to \tilde{A}(C_5) \to H_0^{C_5}(S^{\rho_5 - \rho_5^{\otimes k}}) \to \mathbb{Z} \to 0$$

$H_*^{C_5}(S^{\rho_5-\rho_5^{\otimes k}};A)$

- Our extension problem:

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- If $k = \pm 2 \mod 5$ get a projective $A(C_5)$ module of rank one.

Summary

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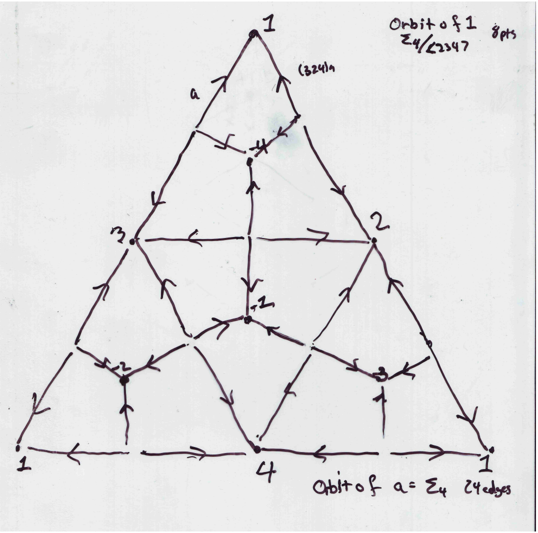
- We can compute the homology and cohomology of these representation spheres.
- Calculating $H_*^G(S^{V+W})$ is generally difficult due to complications in the spectral sequence.
- There is still plenty of other computations left to do.

Twisted tetrahedral representation of Σ_4



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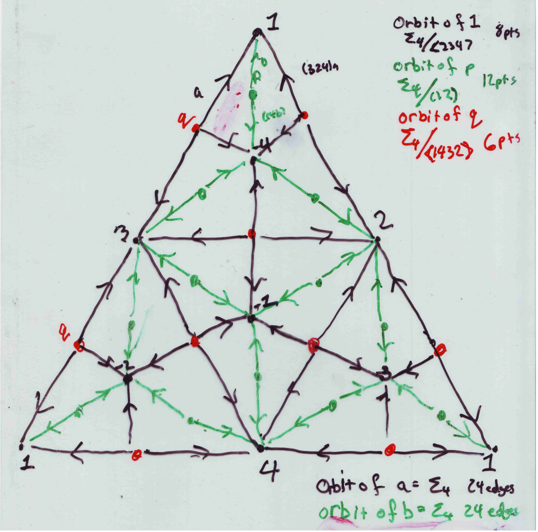

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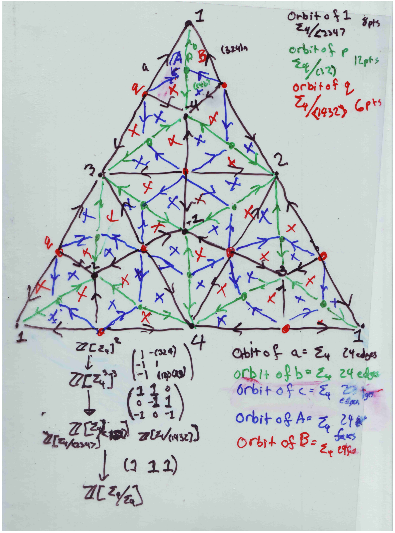

End