### On a nilpotence conjecture of J.P. May

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# $E_{\infty}\overline{/H_{\infty}}$ -rings

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• An  $E_{\infty}$ -ring spectrum R has extended power maps:

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## Examples of $\overline{E_{\infty}}\!/\!H_{\infty}$ -rings

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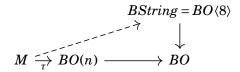
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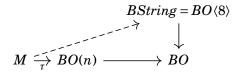
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Applications to bordism

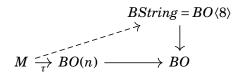
## String manifolds



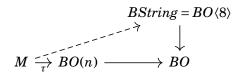
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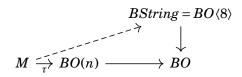


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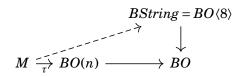
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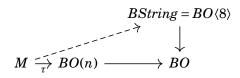
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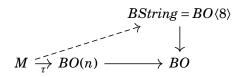


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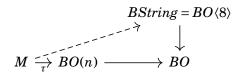
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Applications to bordism

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Applications to bordism

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Applications to bordism

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Applications to bordism

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collapses at  $E_2 \iff$  (all 2-torsion is simple).

 $\bullet \implies$  the BSSs for BString and BSpin collapse at  $E_2$ .

# $H_*(\overline{BString}; \mathbb{Z})$

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Applications to bordism

# $\pi_*MString$

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# End

Thank you for your attention!