Equivariant cohomology and moduli spaces of maps

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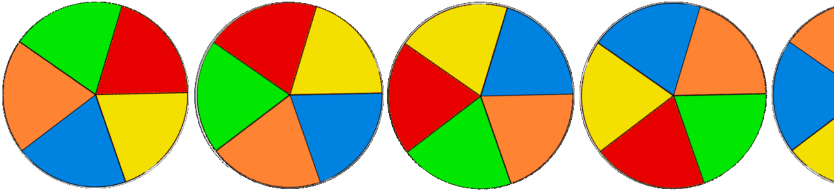
Two projects and why you should care

- Equivariant (co)homology of representation spheres.
  - First computations for non-abelian groups.
  - $\pi_\star H\underline{\mathbb{Z}}$
  - Fun!
- Moduli spaces of maps of algebras.
  - Construct a computational framework for standard questions.
  - Reprove many classic results.
  - Construct new counterexamples.
  - Surprise connection to rational homotopy theory.

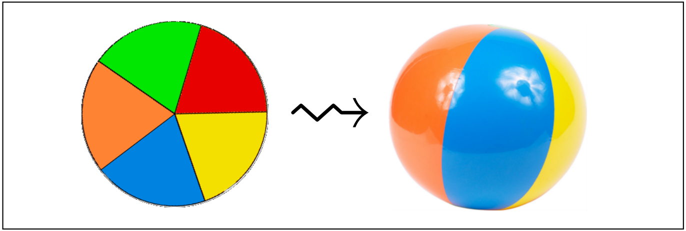
Representation spheres

- Let G be a finite group.
- Take an orthogonal G representation $V = \mathbb{R}^n \circlearrowleft G$.
- Here is a picture of the unit disc of the standard representation of C_5.



G-CW structure

- To construct S^V, collapse the boundary of the unit disc in V to a point.



- Construct a CW-decomposition on S^V, such that G takes cells to cells while never mapping a cell in a non-trivial way to itself.
- For example, the color slices above.

Cellular homology

- Given a CW-complex X let

$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\}$$
$$\widetilde{C}_0(X) := \ker(C_0(X) \to C_0(*))$$

- $H_*(X) \cong \left[\cdots C_{i+1}(X) \xrightarrow{\partial} C_i(X) \xrightarrow{\partial} C_{i-1}(X) \cdots \xrightarrow{\partial} \widetilde{C}_0(X)\right]$
- $H^*(X)$ is calculated by taking the linear dual of this complex and then taking cohomology.

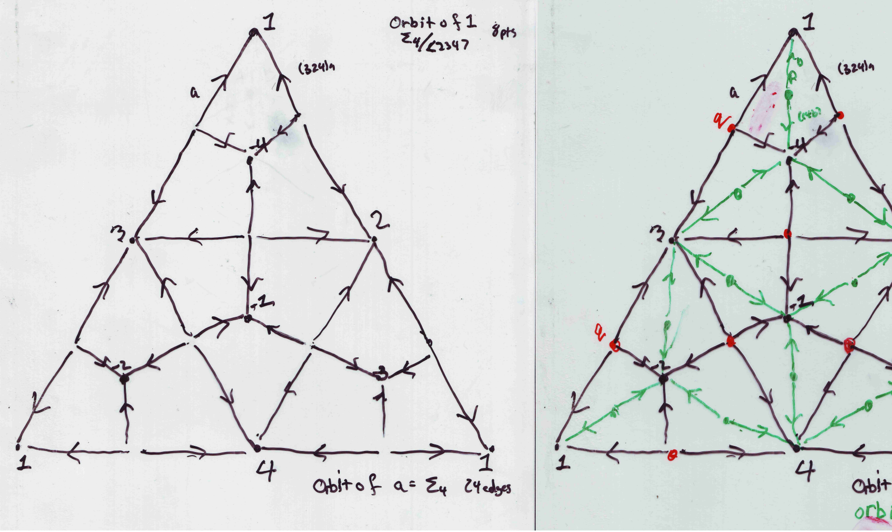
Bredon homology

- Given a G-CW-complex X let
$$C_i(X) := \mathbb{Z}\{i\text{-cells of } X\} \circlearrowleft G.$$
- For a subgroup $K \leqslant G$, let $C_i^K(X) \subset C_i(X)$ be the subgroup of K-invariant chains.
- $H_*^K(X) \cong \left[\cdots C_{i+1}^K(X) \xrightarrow{\partial} C_i^K(X) \xrightarrow{\partial} C_{i-1}^K(X) \cdots \xrightarrow{\partial} \widetilde{C}_0^K(X)\right]$
- When K is the trivial subgroup: $H_*^K(X) \cong H_*(X)$.
- For cohomology first take the invariants on the cochains.

Method of computation

- Fix a finite group G and determine explicit models for all of the irreducible *real* representations of G.
- Construct an explicit G-CW decomposition on each irreducible representation sphere.
- Compute $H_*^G(S^V)$ and $H_G^*(S^V)$.
- Assemble the computations to compute $H_*^G(S^{V\oplus W})$.

Twisted tetrahedral representation of Σ_4



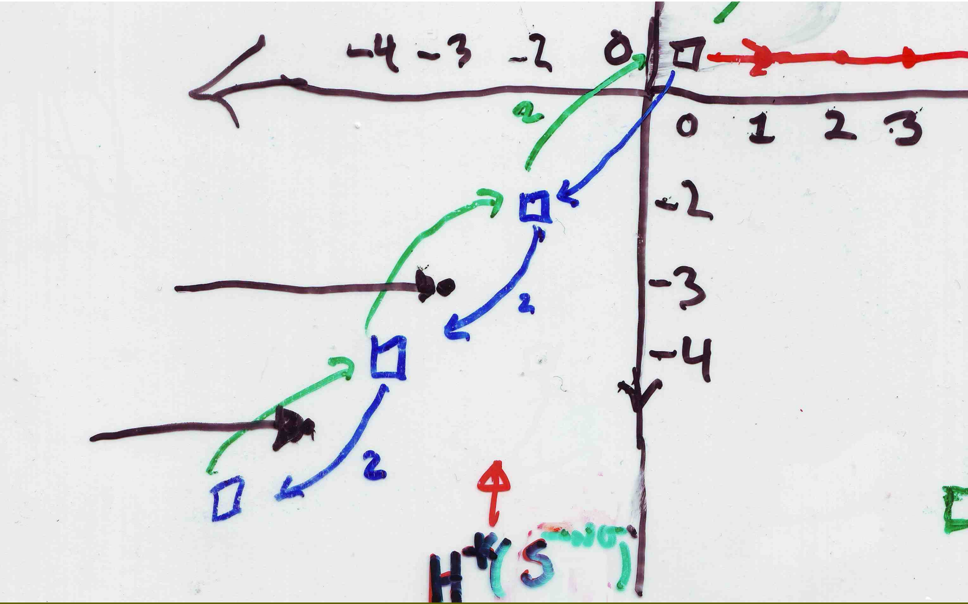
Assembling the computation

- Our main tool for computing the (co)homology of *reducible* representations involves an equivariant Atiyah-Hirzebruch spectral sequence.
- The functor
$$X \mapsto H_*^G(S^W \wedge X)$$
is our generalized equivariant homology theory.
- $$E^1_{*,*} = \text{`}H_*(S^W) \otimes C_*X\text{'} \Longrightarrow H_*^G(S^W \wedge X).$$

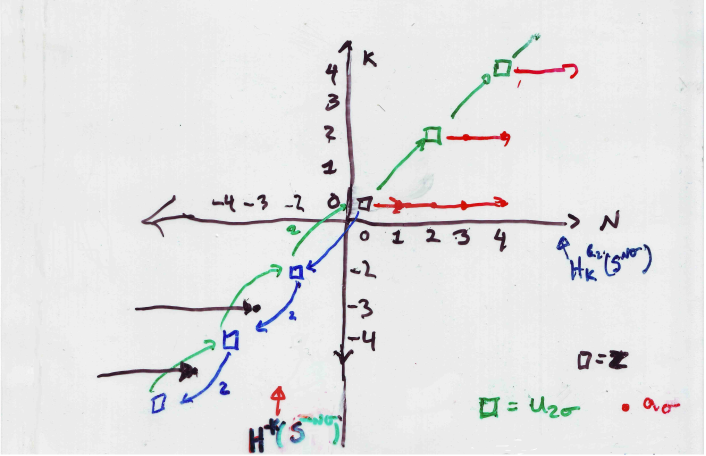
Tricks for computation

1. (Reverse induction) If V is the pullback of a representation of G/H, then use previous computation.
2. (Reciprocity) Use the formula
$$S^W \wedge \mathrm{Ind}_H^G S^i \cong \mathrm{Ind}_H^G \left(\mathrm{Res}_H^G (S^W) \wedge S^i \right)$$
to simplify E_1.
3. (Functoriality) Use subgroup functoriality to determine the differentials.
4. (Competing computations) Decompose the representation in different ways.

Computations for $G = C_2$



Computations for $G = C_2$



Summary

- We can also determine explicit models for the irreducible real representations of C_n, D_n, A_4, and S_4.
- We can compute the homology and cohomology of these representation spheres.
- Calculating $H_*^G(S^{V\oplus W})$ is generally difficult due to complications in the spectral sequence.
- There is still plenty of other computations left to do.

Spaces of maps

Goal

Construct tools for determining $\pi_* \mathscr{C}_T(X,Y)$.

- $\mathscr{C}$ is some category where there is a space of maps between two objects.
- $\mathscr{C}_T$ will be some subcategory whose objects have some additional structure and maps preserve this structure.

To clarify we will consider two examples;
one topological, one algebraic.

First examples

- Consider the functor $X \mapsto \Omega X := \mathcal{T}op_*(S^1, X)$.

Question

Given a map $f \in \mathcal{T}op_*(\Omega X, \Omega Y)$ is $f \sim \Omega g$?

- Suppose A_* and B_* are CDGA's over k $(\text{char } k = 0)$.

Question

Given a map $f \in \mathcal{DG}\text{–}\mathcal{M}od\,(A_*, B_*)$ is $f \sim g$ where g is a map of CDGA's?

First obstructions (Loop spaces)

- Regard ΩX as a monoid by concatenating loops.
- A loop map will always commute with this product structure.
- In particular, the following diagram will commute in the *homotopy category* of based spaces:

$$\begin{tikzcd}
\Omega X \times \Omega X \arrow[r, "f \times f"] \arrow[d] & \Omega Y \times \Omega Y \arrow[d] \\
\Omega X \arrow[r, "f"] & \Omega Y
\end{tikzcd}$$

- Such a map is called an H-map.

First obstructions (Loop spaces)

- We have the following diagram of forgetful functors:

$$\begin{tikzcd}
h\Omega\text{–}\mathcal{T}op(\Omega X,\Omega Y) \arrow[r] & h\mathcal{T}op_*(\Omega X,\Omega Y) \\
 & H\text{–}\mathcal{S}p(\Omega X,\Omega Y) \arrow[d, hook] \\
h\Omega\text{–}\mathcal{T}op(\Omega X,\Omega Y) \arrow[r] \arrow[ur] & h\mathcal{T}op_*(\Omega X,\Omega Y)
\end{tikzcd}$$

- The first obstruction to lifting f to a loop map is to see if it lifts to an H-map.

Question

Are there other obstructions to lifting f to a loop map?Or does this suffice?

First obstructions (CDGAs)

- What are the corresponding notions in the world of CDGAs?
- Note $h\mathcal{DG}\text{–}\mathcal{Mod}(A_*, B_*) \cong \mathcal{Mod}_k(H_*A, H_*B)$.
- The homology of a CDGA is always a graded commutative algebra.
- The H-map analogue is a map $f \in k\text{-}\mathcal{Alg}(H_*A, H_*B)$.

First obstructions (CDGAs)

- We have the following diagram of forgetful functors:

$$\begin{tikzcd}
h\mathcal{CDGA}(A_*,B_*) \arrow[r] & \mathcal{Mod}_k(H_*A,H_*B) \\
 & k\text{-}\mathcal{Alg}(H_*A,H_*B) \arrow[d, hook] \\
h\mathcal{CDGA}(A_*,B_*) \arrow[r] \arrow[ur] & \mathcal{Mod}_k(H_*A,H_*B)
\end{tikzcd}$$

- The first obstruction to lifting f to a map of CDGAs is to see if it lifts to a commutative ring map on homology.

Question

Are there other obstructions to lifting f to a map of CDGAs?Or does this suffice?

Homotopical shadows

- These questions concern whether a map between two objects with structure can be lifted to a map preserving this structure.
- In these cases this structure has some shadow in the homotopy category:
- Loop space $\rightsquigarrow$ H-space (monoid in $h\mathcal{T}op_*$).
- CDGA $\rightsquigarrow$ commutative k-algebra structure in homology.

Spaces of loop maps

Problem in terms of spaces of maps

Compute

$$h\Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y) \cong \pi_0\Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y)$$

and the forgetful maps

$$\pi_0\Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y) \longrightarrow \pi_0\mathcal{T}op_*(\Omega X, \Omega Y)$$

$$\begin{tikzcd} & H\text{–}\mathcal{S}p(\Omega X, \Omega Y) \arrow[d, hook] \\ \pi_0\Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y) \arrow[r] \arrow[ur] & \pi_0\mathcal{T}op_*(\Omega X, \Omega Y) \end{tikzcd}$$

Is the forgetful functor full or faithful here?

Spaces of algebra maps

Problem in terms of spaces of maps

Compute

$$h\mathcal{CDGA}(A_*,B_*) \cong \pi_0 CDGA(A_*,B_*)$$

and the forgetful maps

$$\pi_0\mathcal{CDGA}(A_*,B_*) \longrightarrow \pi_0\mathcal{DG}\text{–}\mathcal{Mod}\,(A_*,B_*)$$

$$\begin{tikzcd} & k\text{-}\mathcal{Alg}(H_*A,H_*B) \arrow[d, hook] \\ \pi_0\mathcal{CDGA}(A_*,B_*) \arrow[ur] \arrow[r] & \pi_0\mathcal{DG}\text{–}\mathcal{Mod}\,(A_*,B_*) \end{tikzcd}$$

Is the forgetful functor full or faithful here?

Monads

- Loop spaces are spaces with additional structure.
- CDGAs are chain complexes with additional structure.
- Both of these structures are examples of T-algebra structures for some monad T.

Spectral sequence

Theorem (Johnson-Noel)

Under technical hypotheses there is a fringed spectral sequence:

$$E_1^{s,t} \Longrightarrow \pi_{t-s}\mathscr{C}_T(X,Y)$$

such that

1. $E_1^{0,0} \cong h\mathscr{C}(X,Y)$
2. $E_2^{0,0} \cong (h\mathscr{C})_T(X,Y)$ *(these correspond to H-maps)*
3. *There are a sequence of obstructions*
$$d_i(f) \in E_1^{i,i-1}$$
to lifting $f \in h\mathscr{C}(X,Y)$ to a map of T-algebras.
4. *The edge homomorphism*
$$\begin{aligned}\pi_0\mathscr{C}(X,Y) \to E_2^{0,0} &\cong (h\mathscr{C})_T(X,Y)\\ \hookrightarrow E_1^{0,0} &\cong h\mathscr{C}(X,Y)\end{aligned}$$
is the previously mentioned forgetful functor.

Applications

- This spectral sequence can be applied to the examples above as well as to a host of other problems.
- Most examples come from algebras over operads.
- We can now show that the homotopy category of E_∞ ring spectra is not equivalent to the category of H_∞ ring spectra.

Example: Hopf map

- Each multiple of the Hopf map $S^3 \to S^2$ defines a map of E_∞ rings $\simeq CDGAs$

$$C^* S^2 \to C^* S^3.$$

- These maps are distinct in $\pi_0 \mathcal{CDGA}(C^* S^2, C^* S^3)$.
- However each induces the trivial map in

$$k\text{-}\mathcal{Alg}(H^* S^2, H^* S^3) \cong \{\varepsilon\}.$$

- This is the first known set of distinct E_∞ maps which induce the same H_∞ map.

Example: Heisenberg manifold

- Let M be the Heisenberg 3-manifold:

$$\begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

- There are infinitely many maps in

$$k\text{-}\mathcal{Alg}(H^* M, H^* S^2)$$

- but $\pi_0 \mathcal{CDGA}(C^* M, C^* S^2) \cong \{\varepsilon\}$.
- This are the first known H_∞ maps which do not lift to E_∞ maps.

Connection to rational homotopy theory

These examples arise for a good reason:

Theorem (Noel)

Suppose X and Y are spaces of finite type and Y is nilpotent then the natural map

$$\mathcal{T}op_*(X, Y_{\mathbb{Q}}) \to E_\infty(H\mathbb{Q}^Y, H\mathbb{Q}^X)$$

is an equivalence.
Moreover this map induces an isomorphism between the classical Bousfield-Kan spectral sequence computing the left hand side and the spectral sequence of Johnson-Noel computing the right hand side.

Final summary

- We computed the equivariant homology and cohomology of some representation spheres.
- We derived an obstruction theoretic spectral sequence solving both old problems and some new ones.
- We used this connection to prove a correspondence between unstable rational homotopy theory and E_∞ ring spectra.

Thank You!