Equivariant cohomology and moduli spaces of maps

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  - Surprise connection to rational homotopy theory.

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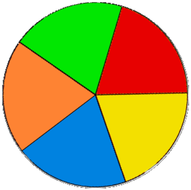

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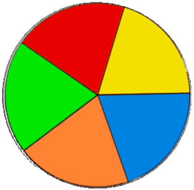

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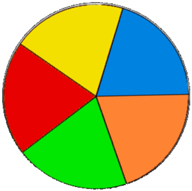

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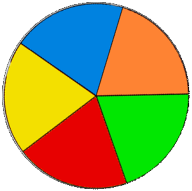

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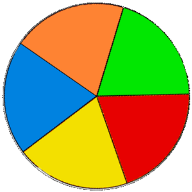

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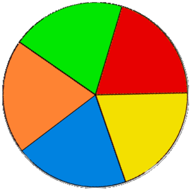

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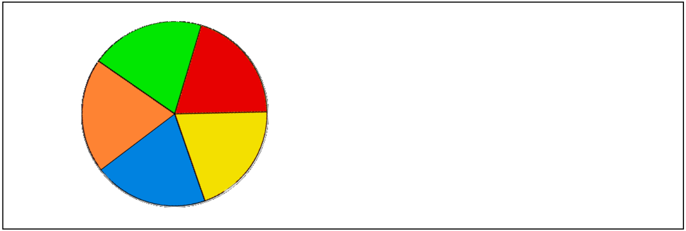

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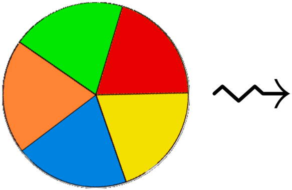

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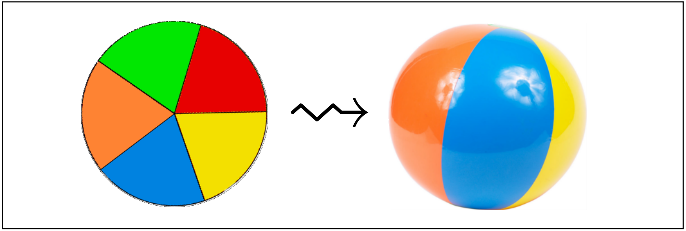

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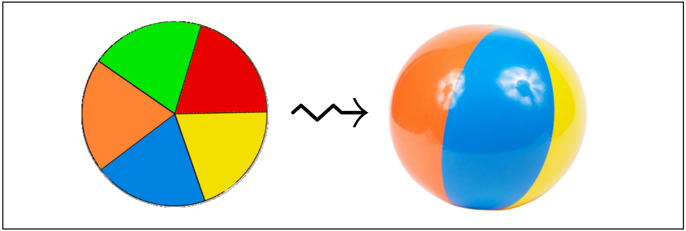

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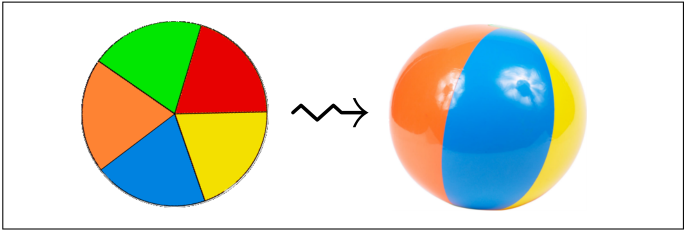

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- For example, the color slices above.

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- $H^*(X)$ is calculated by taking the linear dual of this complex and then taking cohomology.

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- For cohomology first take the invariants on the cochains.

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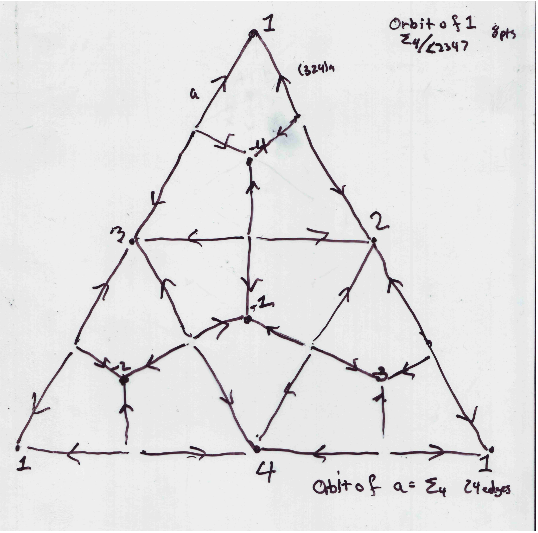
- Compute $H_*^G(S^V)$ and $H_G^*(S^V)$.
- Assemble the computations to compute $H_*^G(S^{V\oplus W})$.

Twisted tetrahedral representation of Σ_4



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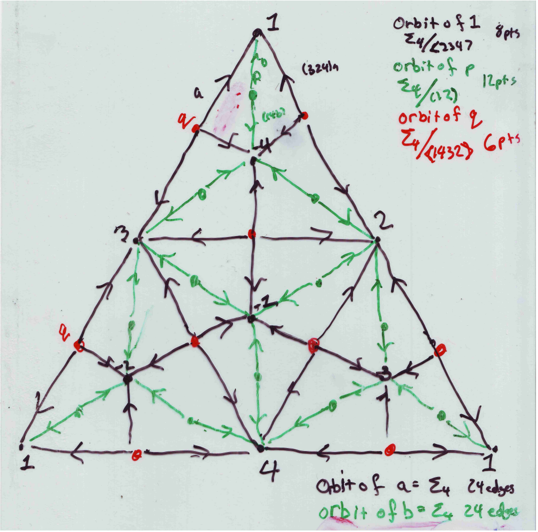

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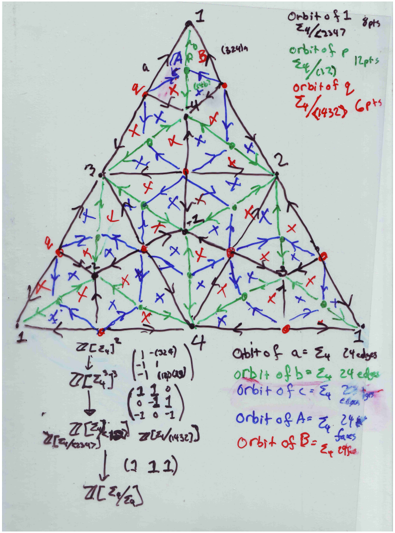

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- The functor
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is our generalized equivariant homology theory.
- $$E^1_{*,*} = \text{`}H_*(S^W) \otimes C_* X\text{'} \Longrightarrow H_*^G(S^W \wedge X).$$

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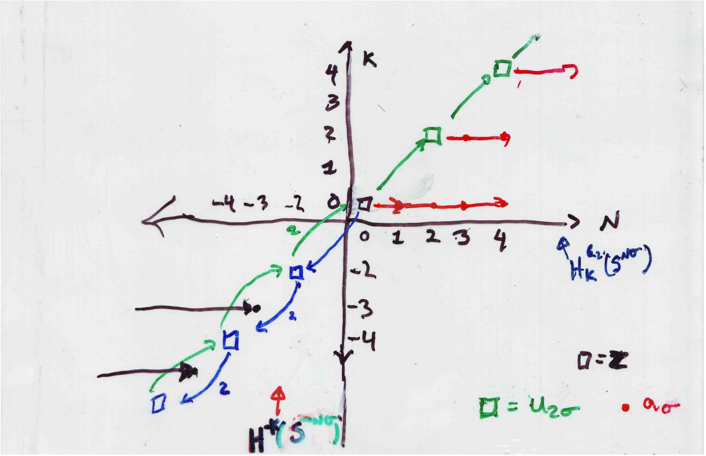
4. (Competing computations) Decompose the representation in different ways.

Computations for $G = C_2$



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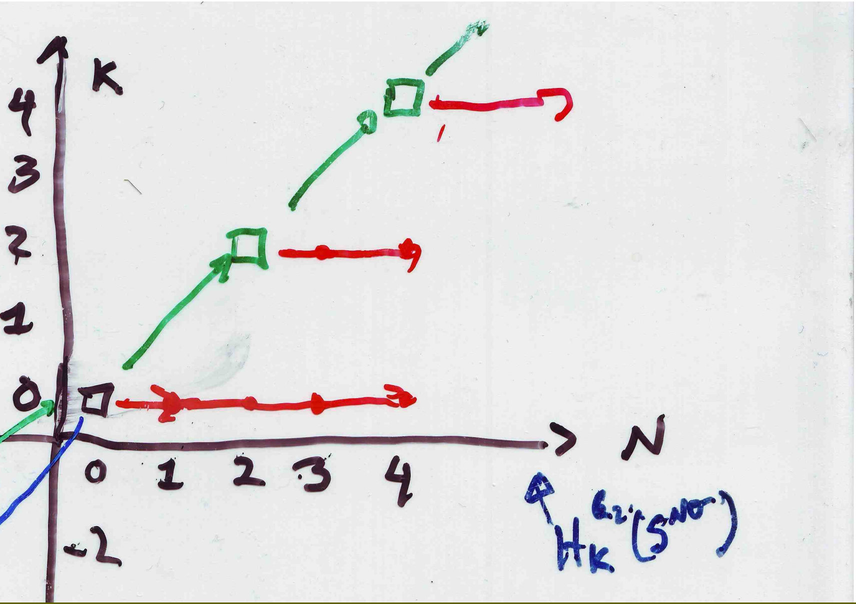

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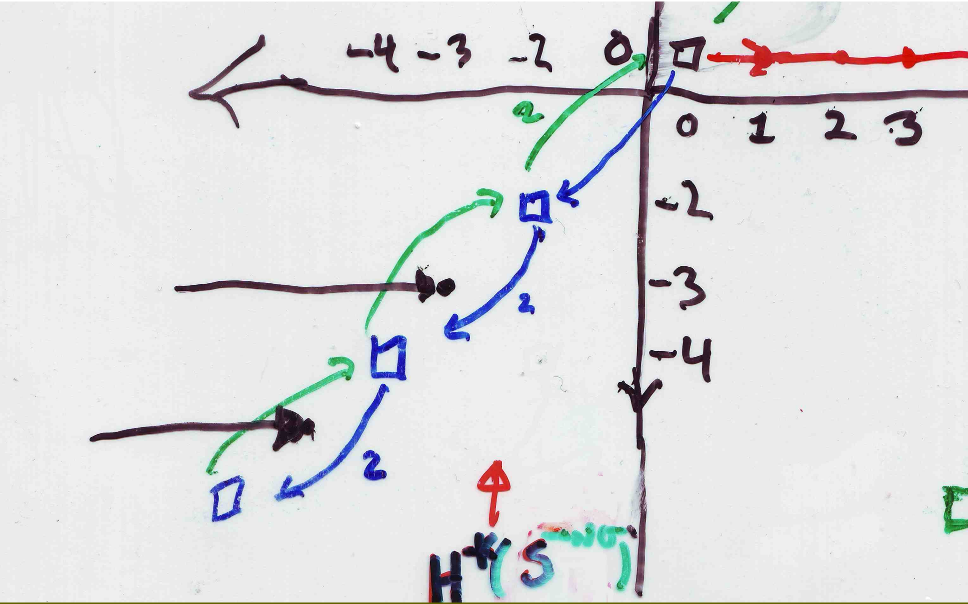

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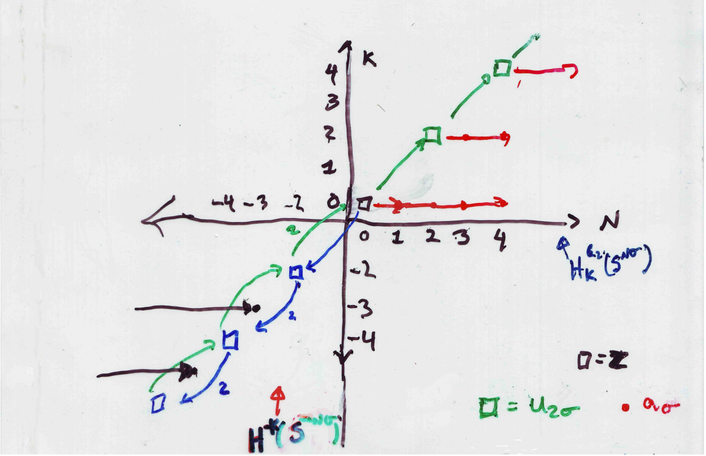

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- Calculating $H_*^G(S^{V\oplus W})$ is generally difficult due to complications in the spectral sequence.
- There is still plenty of other computations left to do.

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- Such a map is called an H-map.

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Are there other obstructions to lifting f to a map of CDGAs?Or does this suffice?

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Spaces of loop maps

Problem in terms of spaces of maps

Compute

$$h\Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y) \cong \pi_0 \Omega\text{–}\mathcal{T}op(\Omega X, \Omega Y)$$

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Is the forgetful functor full or faithful here?

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- Both of these structures are examples of T-algebra structures for some monad T.

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Theorem (Johnson-Noel)

Under technical hypotheses there is a fringed spectral sequence:

$$E_1^{s,t} \Longrightarrow \pi_{t-s}\mathscr{C}_T(X,Y)$$

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- Most examples come from algebras over operads.
- We can now show that the homotopy category of E_∞ ring spectra is not equivalent to the category of H_∞ ring spectra.

Example: Hopf map

- Each multiple of the Hopf map $S^3 \to S^2$ defines a map of E_∞ rings $\simeq CDGAs$

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- This is the first known set of distinct E_∞ maps which induce the same H_∞ map.

Example: Heisenberg manifold

- Let M be the Heisenberg 3-manifold:

$$\begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \Big/ \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}$$

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Moreover this map induces an isomorphism between the classical Bousfield-Kan spectral sequence computing the left hand side and the spectral sequence of Johnson-Noel computing the right hand side.

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Thank You!