The T-algebra spectral sequence: Comparisons and applications

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Spectral sequences as organizational tool

- Spectral sequences are not just computational tools, but also a way to carefully study filtrations.
- EHPSS: Unstable homotopy groups $\Longrightarrow$ stable homotopy groups.
- SS is not efficient for computing $\pi_* S$.
- Filtration info $\rightsquigarrow$ where stable stems are born.
- Does $f : S^{n+k} \to S^n$ lift to non-trivial $[f] \in \pi_k S$?
- If not, what is the obstruction?
- Differentials give obstructions to stability.

Lifting maps of G-spaces

- $X, Y \circlearrowleft G$.
- Can $f \in ho\mathcal{T}op(X,Y)$ be lifted to $ho\mathcal{T}op_G(X,Y)$?
- If not, why not?

$$\begin{tikzcd} ho\mathcal{T}op_G(X,Y) \arrow[rr] \arrow[dr] & & ho\mathcal{T}op(X,Y) \\ & (ho\mathcal{T}op)_G(X,Y) \arrow[ur] & \end{tikzcd}$$

- $(ho\mathcal{T}op)_G \simeq$ spaces with G-actions up to homotopy.

Lifting maps of E_∞ ring spectra

- $X, Y \in E_\infty$ ring spectra ($\mathbb{Q}$-CDGA).
- Can $f \in [X, Y]$ be lifted to $hoE_\infty(X, Y)$?
- If not, why not?

$$\begin{tikzcd}
hoE_\infty(X,Y) \arrow[rr] \arrow[dr] & & {[X,Y]} \\
& H_\infty(X,Y) \arrow[ur] &
\end{tikzcd}$$

- $H_\infty \simeq$ spectra with actions of the E_∞-operad up to homotopy. ($\mathcal{CommAlg}_{\mathbb{Q}}$)

Categorical Setup

- $U : \mathscr{D} \to \mathscr{C}$ monadic. This means:
- U admits left adjoint F.
- $T = UF$ monad on $\mathscr{C}$.

$$\begin{tikzcd} \mathscr{D} \arrow[rr, "U"] \arrow[dr, "\simeq"'] & & \mathscr{C} \\ & \mathscr{C}_T \arrow[ur] & \end{tikzcd}$$

Model Categorical Setup

- Now suppose $U\colon \mathscr{D} \to \mathscr{C}$ simplicial right Quillen.
- $\mathscr{C}_T$ is a simplicial model category.
- Moreover we will ask that U reflects weak equivalences, and commutes with geometric realization of simplicial objects. These are properties we should expect from homotopical monad.
- $\widetilde{T} = \widetilde{U}\widetilde{F}$ monad on $ho\mathscr{C}$.

$$\begin{tikzcd} ho\mathscr{C}_T \arrow[rr] \arrow[dr] & & ho\mathscr{C} \\ & (ho\mathscr{C})_{\widetilde{T}} \arrow[ur] & \end{tikzcd}$$

General question

- Want obstruction theory for lifting maps from $ho\mathscr{C}$ or $(ho\mathscr{C})_{\tilde{T}}$ to $ho\mathscr{C}_T$.
- Better idea:Compute $\pi_*\mathscr{C}_T(X,Y)$ via a spectral sequence where these approximations appear at some stage.
- Differentials should give obstructions.

Note

All of our mapping spaces are derived.

T-algebra spectral sequence

Theorem (Johnson-Noel)

If X is resolvable, then there exists a SS satisfying:

$$E_1^{s,t} \Longrightarrow \pi_{t-s}\mathscr{C}_T(X,Y)$$

such that

1. $E_1^{0,0} \cong ho\mathscr{C}(X,Y)$
2. $E_2^{0,0} \cong (ho\mathscr{C})_{\widetilde{T}}(X,Y)$
3. *Given a basepoint the differentials provide obstructions to lifting $f \in ho\mathscr{C}(X,Y)$ to $ho\mathscr{C}_T$.*
4. *The edge homomorphism*

$$\begin{aligned}\pi_0\mathscr{C}_T(X,Y) \to E_2^{0,0} &\cong (ho\mathscr{C})_{\widetilde{T}}(X,Y)\\ \hookrightarrow E_1^{0,0} &\cong ho\mathscr{C}(X,Y)\end{aligned}$$

is the previously mentioned forgetful functor.

Key points

- Spectral sequence abuts to $\pi_* \mathscr{C}_T(X,Y)$. Not a completion.
- We do *not* require $\mathscr{C}$:
  - to come from an abelian or stable category.
  - to be pointed.
  - to be cofibrantly generated/combinatorial.
  - to be left/right proper.
- We give criteria for the E_2 term to be defined and identifiable with some Quillen cohomology groups.
- In this case we can take our obstructions to lie in the E_2 term.

Resolvability

- X needs to be resolvable with respect to T.
- This implies X is weakly equivalent to an algebra $\tilde{X}$ which is cofibrant in $\mathscr{C}$. But the condition is stronger.
- This happens for any X if
  - $\mathscr{C}_T$ is a category of simplicial algebras in some algebraic category in the sense of Quillen. (Algebra over a multi-sorted Lawvere theory on $s\mathcal{S}et$.)
  - $\mathscr{C}$ is a symmetric monoidal model category compatibly tensored with spaces and $\mathscr{C}_T$ is the category of E_n algebras in $\mathscr{C}$.
  - $\mathscr{C}_T$ is G-objects in $\mathscr{C}$ (naive/genuine variants).

Hirzebruch genera

- G grouplike E_∞-space.
- $f \in E_\infty \mathcal{T}op_*(G, GL_1 S) \rightsquigarrow MG$ is an E_∞ ring spectrum.

Definition

A *Hirzebruch genus* is map of graded commutative rings

$$[\phi]\colon \pi_* MG \to R_*$$

where $R \in \mathcal{C}omm\mathcal{A}lg_{\mathbb{Q}}$.

Lifting Hirzebruch genera

Question

Does there exist an E_∞ ring spectrum R and $\phi \in E_\infty(MG, R)$ such that $\pi_* \phi = [\phi]$?

Answer

Yes. Always. Moreover, this lift is unique up to homotopy through E_∞ maps and genera are in bijection with H_∞ maps.

Note

Note that if $f : G \to GL_1 S$ does not lift to $SL_1 S$ (e.g., $j : O \to GL_1 S$) then $\pi_* MG$ is a $\mathbb{Z}/2$-algebra and there are no genera. So by assumption we have such a lift.

Lifting Hirebruch genera II

- Set $R = HR_*$. The E_∞ Eilenberg-MacLane ring spectrum associated to R_*. This is a choice but the result is independent of the this choice.
- R is $\mathbb{Q}$-local $\implies E_\infty(MG,R) \simeq E_\infty(MG_\mathbb{Q},R)$.
- $\pi_* MG_\mathbb{Q} \cong \pi_* MG \otimes \mathbb{Q}$.
- $[\phi]$ uniquely factors through $\pi_* MG_\mathbb{Q}$.

T-algebra SS for Hirzebruch genera

- $E_2^{0,0} = \mathcal{CommAlg}_{\mathbb{Q}}(\pi_* MG_{\mathbb{Q}}, R_*) \ni [\phi]$.
- $E_2^{s,t} = HAQ^s_{R_*}(\pi_* MG_{\mathbb{Q}}; R_{*+t})$ elsewhere.
- Vanishes for $s > 0$ if $\pi_* MG_{\mathbb{Q}}$ is a free algebra in $\mathcal{CommAlg}_{\mathbb{Q}}$.
- In this case there are *no* obstructions to lifting a genus uniquely and our result holds.

T-algebra SS for Hirzebruch genera

- Now $\pi_* MG_{\mathbb{Q}} \cong H_*(MG;\mathbb{Q}) \cong H_*(BG,\mathbb{Q})$ as graded rings.
- $H_*(BG,\mathbb{Q})$ is a bicommutative connected $\mathbb{Q}$-Hopf algebra.
- By Milnor-Moore this is a free graded commutative $\mathbb{Q}$-algebra.
- And we are done.
- There are similar $E_\infty = H_\infty$ results for computing maps in the $K(n)$-local setting.

Pause

Goerss-Hopkins spectral sequence

- GHSS $\Longrightarrow$ homotopy groups of space of E_∞ maps between spectra.
- Uses Bousfield's E_2/resolution model structure.
- Designed to give computable E_2-term in terms of coalgebras.
- They use that the underlying category is proper, stable, and cellular.

Comparison with GHSS

Proposition

The two spectral sequences agree when we restrict to computing maps of E_∞ algebras in Hk-modules for k a field.

Proof.

Easy direct proof or an easy consequence of an argument due to Jennifer French. □

Flat resolutions in the non-abelian setting

- In E_2/resolution model structure we build resolutions with some designated 'cogroup projectives'.
- French's argument says projective is not necessary. Instead just resolve by objects where the corresponding spectral sequences collapse onto the zero line.
- Analogue of using flat resolutions for Tor calculations.

Some 'flat' resolutions

- The free algebra resolutions appearing in bar resolutions are always 'flat' in Hk-modules $\Longrightarrow$ equivalence between TAlgSS and GHSS in this case.
- Suppose Y is nilpotent of finite type.
- By classical computations and a result of Mandell, the Bousfield-Kan resolutions:

$$H\overline{\mathbb{F}}_p^{\overline{\mathbb{F}}_p^{\cdot+1}Y} \text{ and } H\mathbb{Q}^{\mathbb{Q}^{\cdot+1}Y}$$

will be 'flat' in our examples.

Comparing UASS and GHSS

- This gives a comparison map between

$$\mathrm{UASS} \Longrightarrow \pi_* \mathcal{T}op(X, Y_{\mathbb{Q}})$$

and

$$\mathrm{GHSS} \Longrightarrow \pi_* E_\infty(H\mathbb{Q}^Y, H\mathbb{Q}^X).$$

And their p-primary analogues.

Theorem (French)

These maps induce isomophisms between the UASS and the GHSS.

- In characteristic 0, E_2 term consists of André-Quillen cohomology groups in $\mathcal{C}omm\mathcal{A}lg_{\mathbb{Q}}$.
- In characteristic p, E_2 term consists of Quillen cohomology groups in unstable algebras over the Steenrod algebra.

Using spaces to understand E_∞ ring spectra

Some consequences in TAlgSS/GHSS (Bousfield-Kan)

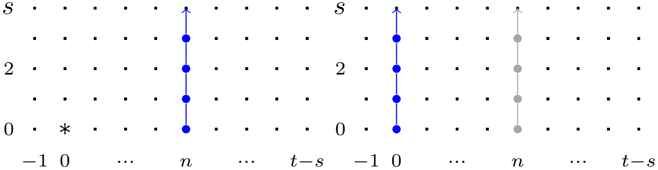
$$\Longrightarrow \pi_* E_\infty(H\mathbb{Q}^Y, H\mathbb{Q}^X)$$

1. Whitehead products in $\pi_* Y$ raise filtration degree.
2. For $t > 0$ and X connected, spectral sequence only depends on $Y_{\mathbb{Q}}$ and $H^*(X;\mathbb{Q})$.
3. Composition product kills elements in positive filtration.
4. Massey products in $H^*(Y;\mathbb{Q})$ give rise to differentials.

Some elementary computations

- Use this to calculate

$$\pi_* K(\mathbb{Z}_p, n) \cong \pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z},n)}, H\overline{\mathbb{F}}_p).$$

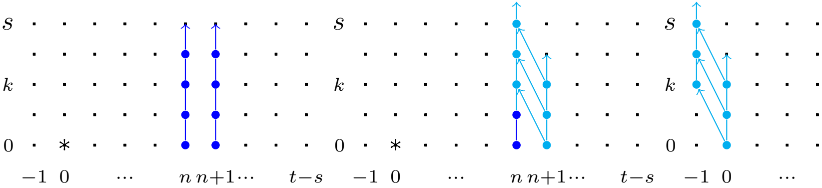


$E_2 = E_\infty$ page for $\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z},n)}, H\overline{\mathbb{F}}_p)$. $E_2 = E_\infty$ page for $\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z},n)}, H\overline{\mathbb{F}}_p^{S^n})$.

- $\implies$ there exists infinitely many distinct E_∞ maps inducing the same H_∞ map (= map of extended DL-algebras).

Some elementary computations II

- Use this to calculate
$$\pi_* K(\mathbb{Z}/p^k, n) \cong \pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z}/p^k,n)}, H\overline{\mathbb{F}}_p)$$
for $k > 1$.



E_2 page for $\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z}/p^k,n)}, H\overline{\mathbb{F}}_p)$ for $k > 1$. E_k and E_∞ pages for $\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z}/p^k,n)}, H\overline{\mathbb{F}}_p)$ for $k > 1$. E_k and E_∞ pages for $\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{K(\mathbb{Z}/p^k,n)}, H\overline{\mathbb{F}}_p^{S^{n+1}})$ for $k > 1$.

- There exists H_∞ maps that are arbitrarily close to E_∞ maps

Bockstein spectral sequence

- We can now construct Bockstein SS topologically.
- Suppose X is a connected space of finite type.
- Calculate
$$\pi_i(\mathrm{Sym}^\infty_* X)_{\widehat{p}} \cong \widetilde{H}_i(X;\mathbb{Z}_p).$$
 via the TAlg/GH/UASS computing
$$\pi_* E_\infty H\overline{\mathbb{F}}_p(H\overline{\mathbb{F}}_p^{\mathrm{Sym}^\infty_* X}, H\overline{\mathbb{F}}_p).$$
- This is the E_2-page of a Bockstein spectral sequence computing the reduced homology of X with $d_r = \beta_r$.
- There is a similar construction for the cohomological Bockstein spectral sequence.

Special cases

To summarize:
The Bockstein SS is a special case of
the unstable Adams SS, which is a special case of
the Goerss-Hopkins SS, which is a special case of
the T-algebra SS.
Through these identifications we obtain information about the filtration between E_∞-algebras and E_∞-algebras up to homotopy.

End